ON THOMPSON'S GROUP *F* AND ITS GROUP ALGEBRA

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Abstract. We have studied about group algebras of non-noetherian groups and showed that they are often primitive if base groups have non-abelian free subgroups. Our main method includes using two edge-colored graphs. In general our method using these graphs seems to be effective for group algebras of groups with non-abelian free subgroups. But there exist some non-Noetherian groups with no non-abelian free subgroups such as Thompson's group F. In this talk, we first introduce an application of (undirected) two edge-colored graphs to group algebras of non-Noetherian groups and then improve our graph theory in order to enable to investigate group algebras of Thompson's group F. Finally, we introduce an application our graph theory to a problem on group algebras of Thompson's group F.

1. Introduction

Let G be a group and KG the group algebra of G over a field K. We denote $KG \setminus \{0\}$, the non-zero elements in KG, by KG^*. KG is a ring which has common right multipliers if for any A and B in KG^*, there exist X and Y in KG^* such that $AX = BY$. We begin with the following simple problem.

Problem 1. *Find elements A and B in KG^* such that $AX + BY \neq 0$ for any X and Y in KG^*. When this is the case, KG does not have common right multipliers.*

If G has a non-abelian free subgroup, then we can find elements A and B of KG^* having the property desired in Problem 1.

In fact, in this case, G has a subgroup freely generated by infinitely many elements; say $a_1, a_2, b_1, b_2, \cdots$. We let here $A = a_1 + a_2$ and $B = b_1 + b_2$ and suppose, to the contrary, that $AX + BY = 0$ for some X and Y in KG^*. Since X and Y in KG, they are expressed as follows:

$$X = \sum_{x \in S_X} \alpha_x x, \quad Y = \sum_{y \in S_Y} \beta_y y,$$

where $\alpha_x, \beta_y \in K \setminus \{0\}$, $S_X = Supp(X)$ and $S_Y = Supp(Y)$. Since $AX + BY = 0$, we have

$$\sum_{x \in S_X} \alpha_x (a_1 x + a_2 x) + \sum_{y \in S_Y} \beta_y (b_1 y + b_2 y) = 0. \tag{1.1}$$

We would like to regard these elements $a_i x$ and $b_i y$ as vertices and construct the graph (V, E, F) with two edge sets E and F. The graph is called a two-edge coloured graph (see the next section). We therefore distinguish all of these elements $a_i x$ and $b_i y$ even

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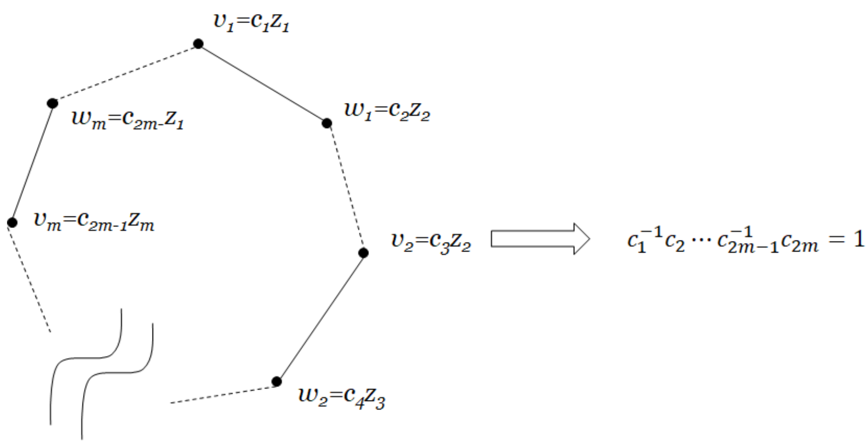
if for $i \neq j$, $a_i x = a_j x'$, $b_i y = b_j y'$ or $a_i x = b_j y$ in G, and define the vertex set as $V = \{(a_i, x), (b_i, y) \mid i = 1, 2, x \in S_X, y \in S_Y\}$. Two edge sets are defined as follows:

$$\begin{array}{lcl} E & = & \{vw \mid v, w \in V; v \neq w, \tilde{v} = \tilde{w} \text{ in } G\}, \text{ where } \tilde{v} = ax \text{ if } v = (a, x). \\ F & = & \{vw \mid v, w \in V; v \neq w, \text{ either } v = (a_1, x), w = (a_2, x) \text{ or } v = (b_1, y), w = (b_2, y)\}. \end{array}$$

Because of (1.1), all elements of G in the left side of the equation (1.1) are cancelled each other. That is, for each $v_1 \in V$, there exists $w_1 \in V$ with $v_1 \neq w_1$ such that $v_1 w_1 \in E$, and then, by the definition of F, there exists $v_2 \in V$ such that $w_1 v_2 \in F$. We can continue with this procedure.



We have $v_1 w_1 \in E$, $w_1 v_2 \in F$, $\cdots$. On the other hand, since V is a finite set, we may assume $z_{m+1} = z_1$, where $v_i = c_{2i-1} z_i$, $w_i = c_{2i} z_{i+1}$, $c_i \in \{a_i, b_i \mid i = 1, 2\}$ and $z_i \in S_X \cup S_Y$. We then get $c_1 z_1 = c_2 z_2$, $c_3 z_2 = c_4 z_3$, $\cdots$. This implies that $c_1^{-1} c_2 \cdots c_{2m-1} c_{2m} = 1$; a contradiction, because $\{a_i, b_i \mid i = 1, 2\}$ is a free basis.

We have thus seen that we can find elements A and B of KG which have the property desired in Problem 1. That is, KG does not have common right multipliers for any field K provided G has a non-abelian free subgroup.

It is known that KG has common right multipliers for any field K provided G is amenable. Therefore we see that G is non-amenable if G has a non-abelian free subgroup. On the other hand, it is an open problem whether Thompson's group F is amenable or not.

The definition of amenability is as follows:

Definition 2. A group G is amenable if for $P(G) = \{S \mid S \subseteq G\}$, there exists $\mu : P(G) \longrightarrow [0, 1]$ such that

1. $\mu(G) = 1$,
2. if S and T are disjoint subsets of G, then $\mu(S \cup T) = \mu(S) + \mu(T)$,
3. if $S \in P(G)$ and $g \in G$, then $\mu(gS) = \mu(S)$.

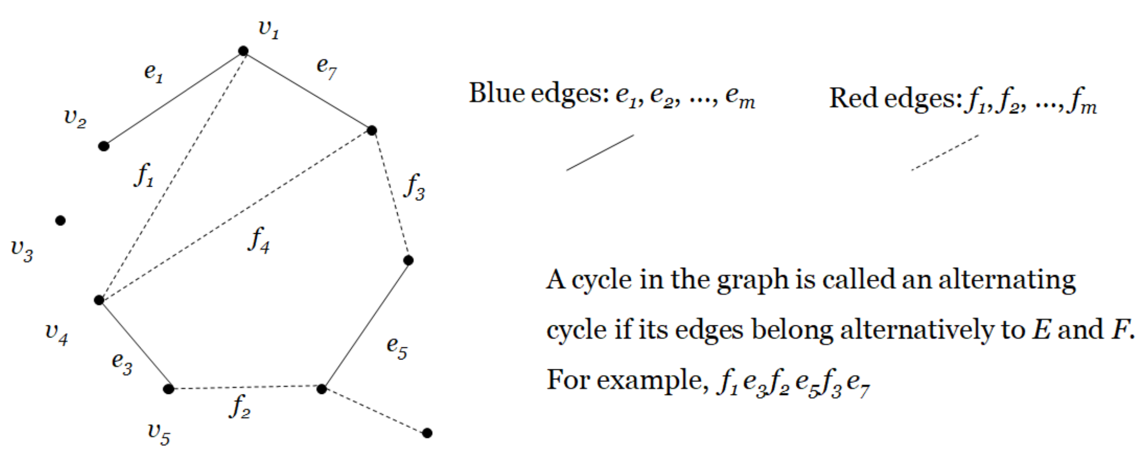
2. SR-graphs

As in the previous section, if G has a free subgroup, then it is easy to find elements A and B of KG such that the right ideal $A(KG) + B(KG)$ generated by A and B is proper;thus for any $X, Y \in KG$, $AX + BY \neq 1$. However, in general, it is often difficult to find elements A_i $(i \in I)$ such that $\sum_{i \in I} A_i X_i \neq 1$ for any $X_i \in KG$ if, for example, A_i has the form $A_i = U_i V_i + 1$ for $V_i, U_i \in KG$. Therefore we need to investigate in which case an SR-cycle exists in an SR-graph with purely graph theoretical consideration.

In this section, we introduce an SR-graph and an SR-cycle; we show that certain SR-graphs have SR-cycles. A class of SR-graphs is a subclass of the class of two-edge coloured graphs which are intensively studied in 1980s and again recently.

Let $\mathcal{G} = (V, E)$ be a simple graph (i.e., an undirected graph without loops or multi-edges) with vertex set V and edge set E. $\mathcal{G}$ is a two-edge coloured graph if each of the edges is coloured either red or blue. We call a path alternating if the successive edges in $\mathcal{G}$ alternate in colour. For any $W \subseteq V$, we let $\mathcal{G}[W]$ denote the subgraph of $\mathcal{G}$ induced by W, i.e., $\mathcal{G}[W] := (W, \{vw \in E \,|\, v, w \in W\})$; let $\mathcal{G}_v := \mathcal{G}[V \setminus \{v\}]$.

A two-edge coloured graph



We let $X(\mathcal{G})$ denote the set of all cut-vertices of $\mathcal{G}$, i.e., the set of all $v \in V$ so that $c(\mathcal{G}_v) > c(\mathcal{G})$. For any terminology and notation which we do not define, we follow [3] (which can also serve as an introductory text if needed).

The following result is due to Grossman and Häggkvist [7]:

Theorem 3. ([7, Theorem]) *Let $\mathcal{G}$ be a two-edge coloured graph so that every vertex is incident with at least one edge of each colour. Then either $\mathcal{G}$ has a cut vertex separating colours, or $\mathcal{G}$ has an alternating cycle.*

We let $I(\mathcal{G})$ denote the isolated vertices of $\mathcal{G}$, i.e., the set of all $v \in V$ for which $vw \notin E$ for all $w \in V$. We denote by $C(\mathcal{G})$ the set of components of $\mathcal{G}$, i.e., the set of subgraphs of $\mathcal{G}$ which partition $\mathcal{G}$, so that in each subgraph any two vertices are joined by a path, and so that no vertices which do not lie in the same subgraph are joined by a path in $\mathcal{G}$; we let $c(\mathcal{G}) := |C(\mathcal{G})|$. We say that $\mathcal{G}$ is connected if $c(\mathcal{G}) = 1$. We begin with two definitions, an SR-graph and an SR-cycle:

Definition 4. Let $\mathcal{G} := (V, E)$ and $\mathcal{H} := (V, F)$. If every component of $\mathcal{G}$ is a complete graph, and if $E \cap F = \emptyset$, then we call the triple $\mathcal{S} = (V, E, F)$ a *sprint relay graph*, abbreviated SR-graph. We view $\mathcal{S}$ as the graph $(V, E \cup F)$, guaranteed simple as $E \cap F = \emptyset$, with edges partitioned into E and F; we denote $\mathcal{S}$ by $(\mathcal{G}, \mathcal{H})$ rather than (V, E, F) when convenient.

Definition 5. A cycle in an SR-graph (V, E, F) is called an SR-cycle if its edges belong alternatively to E and not to E; more formally, we call cycle (V', E') an SR-cycle if there is labeling $V' = \{v_1, v_2, \ldots, v_c\}$ and $E' = \{v_1v_2, v_2v_3, \ldots, v_{c-1}v_c, v_cv_1\}$ so that $v_iv_{i+1} \in E$ if and only if i is odd, for some even c.

Recall that $X(\mathcal{G})$ denote the set of all cut-vertices of $\mathcal{G}$. The following result follows from Theorem 3:

Lemma 6. *If $\mathcal{S}$ has no SR-cycle, then $I(\mathcal{G}) \cup I(\mathcal{H}) \cup X(\mathcal{S}) \neq \emptyset$.*

Let $\mathcal{S} = (V, E, F)$, $\mathcal{G} = (V, E)$, and $\mathcal{H} = (V, F)$ so that $V \neq \emptyset$, every component of $\mathcal{G}$ complete, and $\mathcal{S}$ an SR-graph. Moreover, let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ denote the components of $\mathcal{H}$ with $\mathcal{H}_i = (V_i, E_i)$ over $i \in [n] = \{1, 2, \ldots, n\}$. We first address the case in which $\mathcal{H}_i$ is a complete graph for each $i \in [n]$. By making of Lemma 6 above, we can prove the following theorem:

Theorem 7. ([2, Theorem 2.3]) *If $\mathcal{S}$ is connected and each component of $\mathcal{H}$ is complete, then $\mathcal{S}$ has an SR-cycle if and only if $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$.*

Now, let $I := I(\mathcal{G})$, $W := V \setminus I$, $W_i := V_i \setminus I$, and say $\mathcal{H}[W_i] = (W_i, F_i)$. For any $m_1, m_2, \ldots, m_k \in \mathbb{N}$, we let $K_{m_1,m_2,\ldots,m_k}$ denote the complete multipartite graph with partite sets of size $m_1, m_2, \ldots, m_k$, i.e., the graph (V', E') so that V' can be partitioned into sets $P_1, P_2, \ldots, P_k$ called partite sets, with $|P_i| = m_i$ and $vw \in E'$ if and only if v and w are in different partite sets for all $v, w \in V$. We let $\mu(K_{m_1,m_2,\ldots,m_k}) := \max_{i \in [k]}\{m_i\}$. We now handle the case in which each component of $\mathcal{H}$ is complete multipartite. We can then get the following theorem:

Theorem 8. ([2, Theorem 2.6]) *Assume that $\mathcal{H}_i$ is a complete multipartite graph for each $i \in [n]$. If $|I| \leq n$ and $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in [n]$, then $\mathcal{S}$ has an SR-cycle.*

Theorem 7 and Theorem 8 seem to be effective for the group algebra of a group with a non-abelian free subgroup. In addition, non-Noetherian groups often include non-abelian free subgroups. Therefore, we can show primitivity group algebras for such groups by using these theorems (e.g. [2], [8], [1]). However, there exist some non-Noetherian groups with no non-abelian free subgroups; for example Thompson's group F and a free Burnside group of large exponent. We will next introduce the Thompson's group F and then improve our method to be effective for the group.

3. Thompson's group F

We here briefly introduce the Thompson's group F. We refer the reader to Cannon, Floyd, and Parry [5] for a more detailed discussion of the Thompson's groups (F, T and V).

Originally Thompson's groups $F \subseteq T \subseteq V$ were defined by Richard Thompson in 1965 to construct finitely-presented groups with unsolvable word problems [6]. The Thompson's group F was rediscovered by homotopy theorists in connection with work on homotopy, and then Brin and Squier [4] proved that F does not contain a free group of rank greater than one. After that, many papers on F have been produced until today.

Thompson's group F is defined as a group of piecewise linear maps of the interval $[0, 1]$ as follows:

Definition 9. Thompson's group F is the group (under composition) of those homeomorphisms of the interval $[0, 1]$, which satisfy the following conditions:

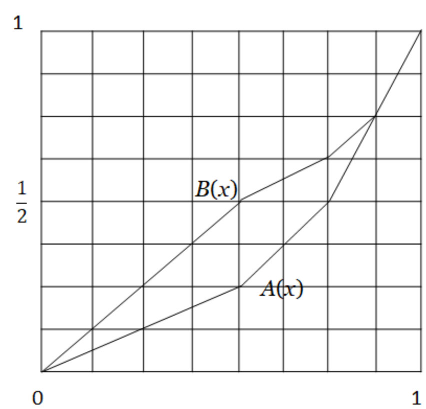
(1) they are piecewise linear and orientation-preserving,

(2) in the pieces where the maps are linear, the slope is always a power of 2, and

(3) the breakpoints are dyadic, i.e., they belong to the set $D \times D$, where $D = [0, 1] \cap \mathbb{Z}[\frac{1}{2}]$.

Example 10. The following two functions A and B are elements in Thompson's group F.

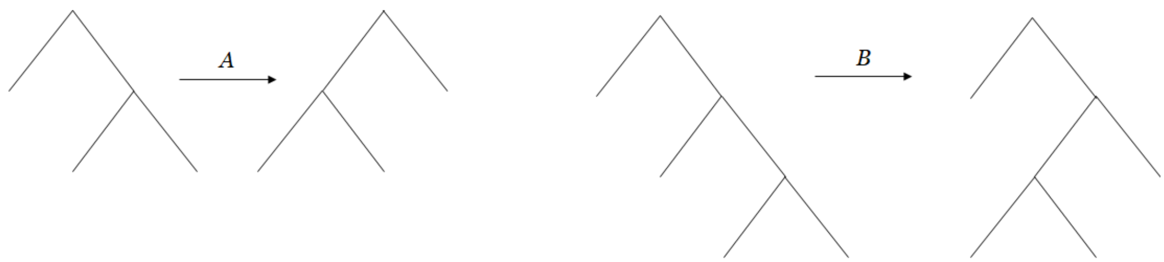
$$A(x) = \begin{cases} \dfrac{x}{2} & 0 \leq x \leq \dfrac{1}{2} \\ x - \dfrac{1}{4} & \dfrac{1}{2} \leq x \leq \dfrac{3}{4} \\ 2x - 1 & \dfrac{3}{4} \leq x \leq 1 \end{cases} \qquad B(x) = \begin{cases} x & 0 \leq x \leq \dfrac{1}{2} \\ \dfrac{x}{2} + \dfrac{1}{4} & \dfrac{1}{2} \leq x \leq \dfrac{3}{4} \\ x - \dfrac{1}{8} & \dfrac{3}{4} \leq x \leq \dfrac{7}{8} \\ 2x - 1 & \dfrac{7}{8} \leq x \leq 1 \end{cases}$$



An element of F can be represented by a tree pair diagram which is a pair of binary trees with the same number of leaves.

Formally, a tree pair diagram is an ordered pair (R, S) of τ-trees such that R and S have the same number of leaves, where τ is defined as follows. The vertices of τ are the standard dyadic intervals in $[0, 1]$. An edge of τ is pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I, J) is a left edge, or I is the right half of J, in which case (I, J) is a right edge.

For example, A and B described above are as follows:



Actually, Thompson's group F is generated by A and B above, and so F is finitely generated. Moreover, F is finitely presented. For example, it is known the following presentation:

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle,$$

where $[x, y]$ denotes the commutator of x and y. On the other hand, F has the following presentation:

$$F = \langle x_0, x_1, x_2, \cdots x_n, \cdots , \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle.$$

For the above presentation, every non-trivial element of F can be expressed in unique normal form

$$x_0^{\beta_0} x_1^{\beta_0} \cdots x_n^{\beta_n} x_n^{-\alpha_n} \cdots x_1^{-\alpha_1} x_0^{-\alpha_0},$$

where n, $\alpha_0, \ldots, \alpha_n, b_0, \cdots, b_n$ are non-negative integers such that

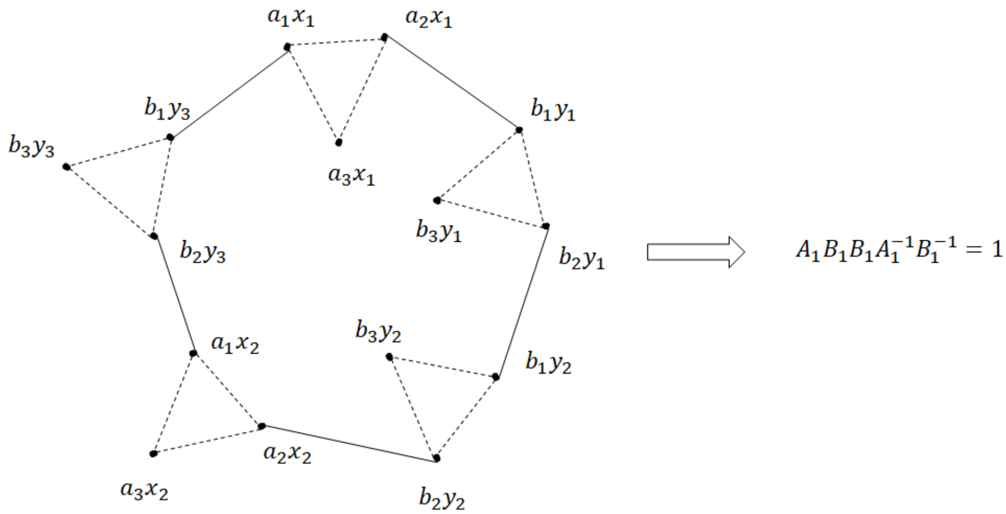
1. exactly one of a_n and b_n is non-zero and
2. if $a_k > 0$ and $b_k > 0$ for some integer k with $0 \leq k < n$, then $a_{k+1} > 0$ or $b_{k+1} > 0$.

As is mentioned above, F is finitely generated and finitely presented. In addition, it is known that F is torsion free and has no non-abelian free subgroup.

4. A directed SR-graph

We first see the following example to know why we need an improvement of SR-graph theory.

Example 11. Let a_i and b_i $(i = 1, 2, 3)$ be in G, and set $A = a_1 + a_2 + a_3$, $B = b_1 + b_2 + b_3$, $A_1 = a_1 a_2^{-1}$ and $B_1 = b_1 b_2^{-1}$. For any $X = \sum_i \alpha_i x_i$ and $Y = \sum_j \beta_j y_j$ in KG^*, We consider the following SR-cycle in an SR-graph:



We have the equation $A_1B_1B_1A_1^{-1}B_1^{-1} = 1$. In general, an SR-cycle in this SR-graph can induce an equation of the form $A_1^{\pm\alpha_1}B_1^{\pm\beta_1}\cdots A_1^{\pm\alpha_m}B_1^{\pm\beta_m} = 1$. Hence, if A_1 and B_1 are free generators in G, then the above equation induce a contradiction. This means that our method is effective for the group algebra of a group with a non-abelian free subgroup.

We would like to improve our method so as to be effective for the group algebra of a group which has no non-abelian free subgroup. To do this, we change a part of an SR-graph which is undirected into a directed graph. We call it a DSR-graph and define as follows:

Definition 12. Let $\mathcal{G} := (V, E)$ and $\mathcal{H} := (V, F)$. If every component of $\mathcal{G}$ is a complete graph, $\mathcal{H}$ is a simple directed graph and if $E \cap F = \emptyset$, then we call the triple $\mathcal{D} = (V, E, F)$ a DSR-graph.

Definition 13. A cycle in an DSR-graph (V, E, F) is called an DSR-cycle if its edges belong alternatively to E and F; more formally, we call cycle (V', E') an DSR-cycle if there is labeling $V' = \{v_1, v_2, \ldots, v_c\}$ and $E' = \{v_1v_2, v_2v_3, \ldots, v_{2m-1}v_{2m}, v_{2m}v_1\}$ so that $v_{2i-1}v_{2i} \in E$ and $(v_{2i}, v_{2i+1}) \in F$.

We might be able to get a desired cycle which induce a equation containing only positive words by using a DSR-graph. This means that our new method does not always need to be a free subgroup in a group. In fact, by making use of our new graph theory, we can get the following result:

Theorem 14. *Let F be a Thompson's group F. If there exist elements a_i, b_i $(i \in [3])$ in F such that for $u_i \in \{a_1a_2^{-1}, a_2a_3^{-1}, a_3a_1^{-1}, b_1b_2^{-1}, b_2b_3^{-1}, b_3b_1^{-1}\}$, $u_1 \cdots u_n = 1$ implies that $u_i \neq c_jc_k^{-1}$ and $u_{i+1} = c_kc_l^{-1}$ for some $i \in [3]$ and $c_i \in \{a_i, b_i \mid i \in [3]\}$, then two elements $A = \sum_{i=1}^{3} a_i$ and $B = \sum_{i=1}^{3} b_i$ of KF satisfy $AX + BY \neq 0$ for any $X, Y \in KG^*$.*

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