GENERAL HEART CONSTRUCTION AND THE GABRIEL-QUILLEN EMBEDDING

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Abstract. The notion of extriangulated category was introduced by Nakaoka and Palu giving a simultaneous generalization of exact categories and triangulated categories. We provide an extension to some extriangulated categories of Auslander's formula, that is, the Serre quotient of the functor category $\mathsf{mod}\,\mathcal{C}$ relative to the Auslander's defects is equivalent to $\mathsf{lex}\,\mathcal{C}$, the full subcategory of left exact functor over $\mathcal{C}$. This is closely related to the Gabriel-Quillen embedding theorem. As an application, we show that the heart of a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category is equivalent to $\mathsf{lex}\,\mathcal{U}$.

Key Words: extriangulated category, Serre quotient, cotorsion pair.

2000 *Mathematics Subject Classification*: Primary 18E10; Secondary 18E30, 18E35.

1. Introduction

Recently, the notion of extriangulated category was introduced in [10] as a simultaneous generalization of triangulated categories and exact categories. It allows us to unify many results on exact categories and triangulated categories in the same framework [4, 8]. A typical example of extriangulated categories (which are possibly neither triangulated nor exact) is an extension-closed subcategory in a triangulated category. Especially, the cotorsion class of a cotorsion pair in a triangulated category has a natural extriangulated structure.

In [2], it was proved that, for any abelian category $\mathcal{A}$, the Yoneda embedding $\mathbb{Y}$ from $\mathcal{A}$ to the category $\mathsf{mod}\,\mathcal{A}$ of finitely presented functors from $\mathcal{A}$ to the category of abelian groups, has an exact left adjoint Q. Moreover the adjoint pair gives rise to a localization sequence

$$\mathsf{def}\,\mathcal{A} \rightleftarrows \mathsf{mod}\,\mathcal{A} \underset{\mathbb{Y}}{\overset{Q}{\rightleftarrows}} \mathcal{A}$$

which is called *Auslander's formula* in [6]. Here $\mathsf{def}\,\mathcal{A}$ denotes the full subcategory of Auslander's defects in $\mathsf{mod}\,\mathcal{A}$ (see Definition 1). The first aim of this article is to present an extension to extriangulated categories of Auslander's formula: for some extriangulated categories $\mathcal{C}$, there exists a localization sequence

$$\mathsf{def}\,\mathcal{C} \rightleftarrows \mathsf{mod}\,\mathcal{C} \underset{R}{\overset{Q}{\rightleftarrows}} \mathsf{lex}\,\mathcal{C}$$

The detailed version of this paper will be submitted for publication elsewhere.

where $\mathsf{lex}\,\mathcal{C}$ denotes the full subcategory of left exact functors in $\mathsf{mod}\,\mathcal{C}$ (Theorem 5). Subsequently, using the composed functor $E_{\mathcal{C}} := Q \circ \mathbb{Y} : \mathcal{C} \to \mathsf{mod}\,\mathcal{C} \to \mathsf{lex}\,\mathcal{C}$, we provide characterizations for a given extriangulated category $\mathcal{C}$ to be exact and abelian, respectively.

Furthermore, considering an expansion of $\mathsf{def}\,\mathcal{C}$ by taking the direct colimits, namely,

$$\overrightarrow{\mathsf{def}\,\mathcal{C}} := \{S \in \mathsf{Mod}\,\mathcal{C} \mid S \text{ is a direct colimit of objects in } \mathsf{def}\,\mathcal{C}\},$$

we construct a localization sequence of $\mathsf{Mod}\,\mathcal{C}$ relative to $\overrightarrow{\mathsf{def}\,\mathcal{C}}$ with a canonical equivalence $\frac{\mathsf{Mod}\,\mathcal{C}}{\overrightarrow{\mathsf{def}\,\mathcal{C}}} \simeq \mathsf{Lex}\,\mathcal{C}$. We explain that, if $\mathcal{C}$ is exact, this localization sequence recovers the Gabriel-Quillen embedding functor (Section 3).

Our second result is an application for a cotorsion pair $(\mathcal{U}, \mathcal{V})$ in a triangulated category $\mathcal{T}$. In [9, 1], it was proved that there exists an abelian category $\underline{\mathcal{H}}$ associated to the cotorsion pair, called the heart, and a cohomological functor $\mathbb{H} : \mathcal{T} \to \underline{\mathcal{H}}$. This result has been shown for two extremal cases [3, 5], namely, t-structures and 2-cluster tilting subcategories (see [9, Proposition 2.6] for details). Since the cotorsion class $\mathcal{U}$ has a natural extriangulated structure, we have thus obtained the localization sequence $\mathsf{def}\,\mathcal{U} \to \mathsf{mod}\,\mathcal{U} \to \mathsf{lex}\,\mathcal{U}$. Using this localization, we provide a good understanding for a construction of the heart and the cohomological functor, especially, there exists an equivalence $\underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{lex}\,\mathcal{U}$.

Notation and convention. For an additive category $\mathcal{C}$, a *(right) $\mathcal{C}$-module* is defined to be a contravariant functor $\mathcal{C} \to \mathsf{Ab}$ and a *morphism* $X \to Y$ between $\mathcal{C}$-modules X and Y is a natural transformation. Thus we define an abelian category $\mathsf{Mod}\,\mathcal{C}$ of $\mathcal{C}$-modules. In the functor category $\mathsf{Mod}\,\mathcal{C}$, the morphism-space $(\mathsf{Mod}\,\mathcal{C})(X, Y)$ is usually denoted by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. We denote by $\mathsf{mod}\,\mathcal{C}$ the full subcategory of finitely presented $\mathcal{C}$-module in $\mathsf{Mod}\,\mathcal{C}$.

2. Auslander's defects over extriangulated categories

Throughout this section, the symbol $\mathcal{C}$ denotes an extriangulated category which admits weak-kernels (see [10] for the definition). We firstly show that the subcategory of defects in $\mathsf{mod}\,\mathcal{C}$ forms a Serre subcategory.

Definition 1. Let $Z \longrightarrow Y \longrightarrow X \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle in an extriangulated category $\mathcal{C}$. Then we have an exact sequence $(-, Z) \to (-, Y) \to (-, X) \to \tilde{\delta} \to 0$ in $\mathsf{mod}\,\mathcal{C}$. The functor $\tilde{\delta}$ is called a *defect of δ*. We denote by $\mathsf{def}\,\mathcal{C}$ the full subcategory in $\mathsf{mod}\,\mathcal{C}$ consisting of all functors isomorphic to defects.

This notion was originally introduced by Auslander in the case that $\mathcal{C}$ is abelian.

Proposition 2. *Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then, the subcategory $\mathsf{def}\,\mathcal{C}$ forms a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$.*

Thus we have a Serre quotient of $\mathsf{mod}\,\mathcal{C}$ relative to $\mathsf{def}\,\mathcal{C}$. We consider the following perpendicular category

$$(\mathsf{def}\,\mathcal{C})^{\perp} := \{F \in \mathsf{mod}\,\mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(G, F) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(G, F) \text{ for any } G \in \mathsf{def}\,\mathcal{C}\}.$$

To understand the Serre quotient, it is basic to study the perpendicular category. The following proposition shows that $(\mathsf{def}\,\mathcal{C})^{\perp}$ coincides with the full subcategory of left exact functors.

Definition 3. Let $\mathcal{A}$ and $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an abelian category and an extriangulated category, respectively. A contravariant functor $F:\mathcal{C}\to\mathcal{A}$ is said to be *left exact*, if F sends a conflation $Z\to Y\to X$ to an exact sequence $0\to FZ\to FY\to FX$. We denote by $\mathsf{lex}\,\mathcal{C}$ (resp. $\mathsf{Lex}\,\mathcal{C}$) the full subcategory of all left exact functors in $\mathsf{mod}\,\mathcal{C}$ (resp. $\mathsf{Mod}\,\mathcal{C}$).

Let us remark that, if $\mathcal{C}$ is a triangulated category, the left exact functors should be zero.

Proposition 4. *Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then, we have an equality $(\mathsf{def}\,\mathcal{C})^{\perp}=\mathsf{lex}\,\mathcal{C}$.*

The following is our first result which directly follows from Propositions 2 and 4.

Theorem 5. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category with weak-kernels. Then, we have a Serre quotient*

$$\mathsf{def}\,\mathcal{C}\longrightarrow\mathsf{mod}\,\mathcal{C}\xrightarrow{\;Q\;}\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}. \tag{2.1}$$

Moreover, if the quotient functor Q has a right adjoint, we have a localization sequence

$$\mathsf{def}\,\mathcal{C}\rightleftarrows\mathsf{mod}\,\mathcal{C}\underset{R}{\overset{Q}{\rightleftarrows}}\mathsf{lex}\,\mathcal{C} \tag{2.2}$$

where R deotes the canonical inclusion.

If $\mathcal{C}$ is abelian, the above localization sequence (2.2) is nothing other than the following Auslander's formula ([2, p. 205]).

Proposition 6 (Auslander's formula)**.** *Suppose that $\mathcal{C}$ is abelian. Then, the Yoneda embedding $\mathbb{Y}:\mathcal{C}\hookrightarrow\mathsf{mod}\,\mathcal{C}$ admits an exact left adjoint Q. Moreover, we have a localization sequence:*

$$\mathsf{def}\,\mathcal{C}\rightleftarrows\mathsf{mod}\,\mathcal{C}\underset{\mathbb{Y}}{\overset{Q}{\rightleftarrows}}\mathcal{C}.$$

In particular, Auslander's formula and Theorem 5 tell us that, for abelian category $\mathcal{C}$: (1) the subcategory $\mathsf{def}\,\mathcal{C}$ is localizing, namely, the associated quotient functor $Q:\mathsf{mod}\,\mathcal{C}\to\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$ has a right adjoint; (2) there exists an equivalence $\mathcal{C}\simeq\mathsf{lex}\,\mathcal{C}$. However, even if a given category $\mathcal{C}$ is exact, the quotient functor Q does not necessarily have a right adjoint (see [11, Example 2.10]).

The following theorem provides characterizations for $\mathcal{C}$ to be exact or abelian via the functor $E_{\mathcal{C}}:=Q\mathbb{Y}:\mathcal{C}\to\frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$.

Theorem 7. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category with weak-kernels. Then the following hold.*

1. *The functor $E_{\mathcal{C}}$ is exact and fully faithful if and only if $\mathcal{C}$ is an exact category.*
2. *The functor $E_{\mathcal{C}}$ is an exact equivalence if and only if $\mathcal{C}$ is an abelian category. If this is the case, we have an equivalence $\mathcal{C}\simeq\mathsf{lex}\,\mathcal{C}$.*

2.1. **The case of enough projectives.** We study the case that an extriangulated category $\mathcal{C}$ has enough projectives.

Definition 8. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. We say that $\mathcal{C}$ *has enough projectives* if there exists a full subcategory $\mathcal{P}$ in $\mathcal{C}$ with $\mathbb{E}(\mathcal{P}, \mathcal{C}) = 0$ and, for every $C \in \mathcal{C}$, there exists a conflation $C' \to P \to C$ with $P \in \mathcal{P}$.

In this case, we have nicer forms of the quotient functor $Q : \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$ and the functor $E_{\mathcal{C}} : \mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$.

Proposition 9. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category with weak-kernels which has enough projectives. Let $\mathcal{P}$ be the subcategory of projectives in $\mathcal{C}$ and consider the restriction functor $\mathsf{res}_{\mathcal{P}} : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$. Then the following hold.*

1. *There exists an equivalence $Q' : \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \simeq \mathsf{mod}\,\mathcal{P}$ with $\mathsf{res}_{\mathcal{P}} \cong Q' \circ Q$.*
2. *The functor $E_{\mathcal{C}} : \mathcal{C} \to \mathsf{mod}\,\mathcal{P}$ sends X to $\mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{P}}$, where $\mathrm{Hom}_{\mathcal{C}}(-, X)|_{\mathcal{P}}$ is a restricted functor on $\mathcal{P}$.*
3. *An equality $\mathsf{def}\,\mathcal{C} = \mathsf{mod}(\mathcal{C}/[\mathcal{P}])$ holds in $\mathsf{mod}\,\mathcal{C}$.*

We end this section by mentioning that, in the case that $\mathcal{C}$ is an exact category having enough projectives, the quotient functor $Q : \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \simeq \mathsf{mod}\,\mathcal{P}$ always admits a right adjoint.

Proposition 10. *Let $(\mathcal{C}, \mathbb{E})$ be an exact category with weak-kernels which has enough projectives. Then, the restriction functor $\mathsf{res}_{\mathcal{P}} : \mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{P}$ admits a right adjoint R. Moreover, it induces a recollement*

$$\begin{tikzcd} \mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\mathsf{res}_{\mathcal{P}}"] & \mathsf{mod}\,\mathcal{P} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

3. Connection to the Gabriel-Quillen embedding theorem

In this section, we study a connection between the localization sequence (2.2) and the Gabriel-Quillen embedding theorem. Let $\mathcal{C}$ be a skeletally small extriangulated category with weak-kernels. We denote by $\overrightarrow{\mathsf{def}\,\mathcal{C}}$ the full subcategory in $\mathsf{Mod}\,\mathcal{C}$ consisting of direct colimits of objects in $\mathsf{def}\,\mathcal{C}$.

Theorem 11. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a skeletally small extriangulated category with weak-kernels. Then, the Serre quotient (2.1) induces the following localization sequence*

$$\begin{tikzcd} \overrightarrow{\mathsf{def}\,\mathcal{C}} \arrow[r] & \mathsf{Mod}\,\mathcal{C} \arrow[l, bend left] \arrow[r] & \mathsf{Lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd} \tag{3.1}$$

where R denotes the canonical inclusion. Moreover, the composed functor $\mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C} \to \mathsf{Lex}\,\mathcal{C}$ is isomorphic to the Gabriel-Quillen embedding functor.

4. General heart construction versus Left exact functors

Throughout this section, we fix a triangulated category $\mathcal{T}$ with a translation [1]. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in $\mathcal{T}$ (equivalently, $(\mathcal{U}, \mathcal{V}[1])$ forms a torsion pair in $\mathcal{T}$). Since $\mathcal{U}$ is extension-closed and contravariantly finite in $\mathcal{T}$, it gives rise to an extriangulated category with weak-kernels by setting $\mathbb{E}(+,-) := \mathcal{U}(+,-[1])$. For this extriangulated category $\mathcal{U}$, the associated quotient functor $Q : \mathsf{mod}\,\mathcal{U} \to \frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}}$ has a right adjoint.

Proposition 12. *The quotient functor $Q : \mathsf{mod}\,\mathcal{U} \to \frac{\mathsf{mod}\,\mathcal{U}}{\mathsf{def}\,\mathcal{U}}$ has a right adjoint. Moreover, there exists a localization sequence*

$$\mathsf{def}\,\mathcal{U} \longrightarrow \mathsf{mod}\,\mathcal{U} \xrightarrow{Q} \mathsf{lex}\,\mathcal{U}, \qquad \mathsf{def}\,\mathcal{U} \longleftarrow \mathsf{mod}\,\mathcal{U} \xleftarrow{R} \mathsf{lex}\,\mathcal{U}$$

where R denotes the canonical inclusion.

Finally we study a connection between $\mathsf{lex}\,\mathcal{U}$ and the heart of the cotorsion pair $(\mathcal{U}, \mathcal{V})$. Let us introduce the following notion: For two classes $\mathcal{U}$ and $\mathcal{V}$ of objects in $\mathcal{T}$, we denote by $\mathcal{U} * \mathcal{V}$ the class of objects X occurring in a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Definition 13. Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$. We define the following associated categories:

- Put $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$;
- For a sequence $\mathcal{W} \subseteq \mathcal{S} \subseteq \mathcal{T}$ of subcategories, we put $\underline{\mathcal{S}} := \mathcal{S}/[\mathcal{W}]$ and denote by $\pi : \mathcal{S} \to \underline{\mathcal{S}}$ the canonical ideal quotient functor;
- We put $\mathcal{T}^+ := \mathcal{W} * \mathcal{V}[1], \mathcal{T}^- := \mathcal{U}[-1] * \mathcal{W}$ and $\mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-$.

We call the category $\underline{\mathcal{H}}$ the *heart of* $(\mathcal{U}, \mathcal{V})$.

As mentioned in Introduction, the heart $\underline{\mathcal{H}}$ is abelian and there exists a *cohomological* functor $\mathbb{H} : \mathcal{T} \to \underline{\mathcal{H}}$, namely, $\mathbb{H}$ sends any triangle $X \to Y \to Z \to X[1]$ in $\mathcal{T}$ to an exact sequence $\mathbb{H}X \to \mathbb{H}Y \to \mathbb{H}Z \to \mathbb{H}X[1]$ in $\underline{\mathcal{H}}$. The following provides us a good understanding for the heart $\underline{\mathcal{H}}$ and the cohomological functor $\mathbb{H}$.

Theorem 14. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$. Then the following hold.*

(1) *There exists a natural equivalence $\Psi : \underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{lex}\,\mathcal{U}[-1]$.*

(2) *The cohomological functor $\mathbb{H}$ is isomorphic to the composed functor $\mathcal{T} \to \mathsf{mod}\,\mathcal{U}[-1] \xrightarrow{Q} \mathsf{lex}\,\mathcal{U}[-1] \xrightarrow{\Psi^{-1}} \underline{\mathcal{H}}$.*

The construction of the equivalence $\Psi : \underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{lex}\,\mathcal{U}[-1]$ is as follows: By Proposition 12, we have a localization sequence of $\mathsf{mod}\,\mathcal{U}[-1]$ relative to $\mathsf{def}\,\mathcal{U}[-1]$. We consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{H} & \hookrightarrow & \mathcal{T} & \xrightarrow{\mathbb{Y}_{\mathcal{U}[-1]}} & \mathsf{mod}\,\mathcal{U}[-1] & \xrightarrow{Q} & \mathsf{lex}\,\mathcal{U}[-1] \\ \downarrow{\scriptstyle\pi} & & & & & \nearrow{\scriptstyle\Psi} & \\ \underline{\mathcal{H}} & & & & & & \end{array}$$

There uniquely exists a dotted arrow Ψ which makes the diagram commutative up to isomorphism. Hence, we have an isomorphism $\Psi(\pi(H)) \cong \mathrm{Hom}_{\mathcal{T}}(-, H)|_{\mathcal{U}[-1]}$ for each $H \in \mathcal{H}$, which gives an explicit description of the equivalence Ψ.

Theorem 14 generalize the following result.

Corollary 15. [7, Thm. 2.10] *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair in a triangulated category $\mathcal{T}$ and $\mathcal{P}$ the full subcategory of projectives in the extrianguated category $\mathcal{U}$. If $\mathcal{U}$ has enough projectives, then we have an equivalence $\underline{\mathcal{H}} \xrightarrow{\sim} \mathsf{mod}\,\mathcal{P}$.*

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