ON LIFTABLE DG MODULES
OVER A COMMUTATIVE DG ALGEBRA

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Abstract. Let A be a DG algebra and $B = A\langle X \mid dX = t\rangle$ be an extended DG algebra of A by the adjunction of a variable of positive even degree n. Let N be a semi-free DG B-module that is assumed to be bounded below as a graded module. In this article, we discuss a lifting problem for N in the situation $A \to B$. We explain how to construct an obstruction for lifting N to A as an element of $\mathrm{Ext}_B^{n+1}(N, N)$.

1. Introduction

This report is based on a joint work with Yuji Yoshino[7].

M. Auslander, S. Ding and Ø. Solberg [1] studied liftings and weak liftings of finitely generated modules over a commutative Noetherian algebra. Recently, S. Nasseh and S. Sather-Wagstaff [5], and S. Nasseh and Y. Yoshino [6] extended them to the case of DG modules over commutative DG algebras.

We fix a commutative ring R. Let A be a commutative DG R-algebra and X be a variable of degree $|X|$. Then one can construct $B = A\langle X \mid dX = t\rangle$ denotes an extended DG R-algebra by adding the variable X with relation $dX = t$. See §2 below for more details. There is a natural DG algebra homomorphism $A \to B$.

We concern a lifting problem for $A \to B = A\langle X \mid dX = t\rangle$. In the both papers[5, 6], they only considered the lifting problem in such cases but with the assumption that $|X|$ is *odd*. In this case, B is a Koszul complexes of A. They actually construct an obstruction for weakly lifting a semi-free DG B-module N to A as an element of $\mathrm{Ext}^{|X|+1}(N, N)$.

In contrast, our main target in the present article is the lifting problem for $A \to B = A\langle X \mid dX = t\rangle$ where $|X|$ is positive and *even*. In this case, B is is a free algebra over A with a divided powers variable X that resemble a polynomial ring over A. Let N be a semi-free DG B-module that is assumed to be bounded below as a graded module. The aim of this article is to explain how to construct an obstruction for lifting N to A as an element of $\mathrm{Ext}_B^{|X|+1}(N, N)$. To do this, we introduce a certain operator on the set of graded R-linear endomorphisms on N, which is called the *j-operator*. Furethermore, we prove that such a lifting module is unique up to DG A-isomorphisms if $\mathrm{Ext}_B^{|X|}(N, N) = 0$.

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2. DG algebras and DG modules

In this article, R always means a commutative ring. All DG R-algebra in this article are meant to be a commutative DG algebra over R. We omit definitions of a DG R-algebra and a DG module; see [2, 4].

In this section, we summarize some materials which we will use in the next section. For a DG R-algebra A and a DG A-module M, we often denote by $A^\natural$ the underlying graded R-algebra of A and by $M^\natural$ the underlying graded $A^\natural$-module of M.

Let M and N be a DG module over a DG R-algebra A. We define the DG A-module $\mathrm{Hom}_A(M,N)$ as $\mathrm{Hom}_A(M,N)^\natural = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Grmod}A^\natural}(M^\natural, N^\natural(n))$ where $\mathrm{Grmod}A^\natural$ denotes the category of graded $A^\natural$-modules and $N^\natural(n)$ denotes the twist of N by n. The differential on $\mathrm{Hom}_A(M,N)$ is given by

$$\partial^{\mathrm{Hom}_A(M,N)}(f) = \partial^N \circ f - (-1)^{|f|} f \circ \partial^M$$

where f is a homogeneous $A^\natural$-linear homomorphism and $|f|$ denotes the degree of f.

A DG A-module F is said to be *semi-free* if $F^\natural$ has a graded $A^\natural$-free basis E which decomposes as a disjoint union $E = \bigsqcup_{i\geqq 0} E_i$ and satisfies $\partial^F(E_i) \subseteq \sum_{j<i} AE_j$ for $i \geqq 0$. For a semi-free DG A-module F and an integer n, we define the n-th self extension module by

$$\mathrm{Ext}^n_A(F,F) := H_{-n}(\mathrm{Hom}_A(F,F)).$$

One can define the n-th extension module $\mathrm{Ext}^n_A(M,N)$ for arbitrary DG A-modules M and N by a different condition. See [3] for more detail.

Let $A \to B$ be a DG algebra homomorphism and M be a DG A-module. The DG A-module $B \otimes_A M$ is defined as follows: the underling module $(B \otimes_A M)^\natural$ is the tensor product $B^\natural \otimes_{A^\natural} M^\natural$ of graded $A^\natural$-modules and its differential is given by

$$\partial^{B\otimes_A M}(b \otimes m) = d^B(b) \otimes m + (-1)^{|b|} b \otimes \partial^M(m)$$

where b is a homogeneous element in B and $|b|$ denotes the degree of b. Then $B \otimes_A M$ is regarded as a DG B-module via action $b(b' \otimes m) = bb' \otimes m$ for $b, b' \in B$ and $m \in M$.

Definition 1. Let $A \to B$ be a DG algebra homomorphism.

(1) A semi-free DG B-module N is *liftable* to A if there is a semi-free DG A-module M such that $N \cong B \otimes_A M$ as DG B-modules. In this case, M is called a lifting of N to A.
(2) A semi-free DG B-module N is *weakly liftable* to A if there is a semi-free DG A-module M such that N is a direct summand of the DG B-module $B \otimes_A M$.

The following DG algebras are main objects of this article. Let A be a DG R-algebra and t be a cycle in A, i.e. $d^A(t) = 0$. We construct an extended DG algebra B of A by the adjunction of a variable X with $|X| = |t| + 1$ to kill the cycle t in the following way. See [2, 4, 8] for details. In both cases, we denote B by $A\langle X \mid dX = t\rangle$.

(1) If $|X|$ is odd, then $B^\natural = A^\natural \oplus XA^\natural$ is the graded free $A^\natural$-module with basis $\{1, X\}$ and with a multiplication structure: $(a + Xb)(a' + Xb') = aa' + X(ba' + (-1)^{|a|}ab')$ for $a, b, a', b' \in A$. The differential on B is defined by $d^B(a + Xb) = d^A(a) + tb - Xd^A(b)$ for $a, b \in A$.

(2) If $|X|$ is even, then $B^\natural = \bigoplus_{i\geqq 0} X^{(i)}A^\natural$ is the graded free $A^\natural$-module with basis $\{X^{(i)} : |X(i)| = i|X|\}_{i\geqq 0}$ and with a multiplication rule $X^{(i)}X^{(j)} = \binom{i+j}{j}X^{(i+j)}$ for $i, j \in \mathbb{Z}$. Here we use the convention $X^{(0)} = 1$, $X^{(1)} = X$. The differential on B is defined by $d^B(X^{(i)}) = tX^{(i-1)}$ for $i \geqq 1$.

In each case, there is a natural DG R-algebra homomorphism $A \to B = A\langle X \mid dX = t\rangle$.

As we have mentioned in the introduction, we concern the lifting problem in the situation $A \to B = A\langle X \mid dX = t\rangle$ where $|X|$ is even. Recently, S. Nasseh and Y. Yoshino have studied a weakly liftable condition for semi-free DG B-modules in the case where $|X|$ is odd. See [6, Theorem 3.6].

3. Main Results

We begin by establishing some notation to be used in this section.

Notation 2. Let A be a DG R-algebra and t be a cycle in A of odd degree. We denote by $B = A\langle X \mid dX = t\rangle$ an extended DG algebra of A by the adjunction of a variable X that kills the cycle t. Note that $|X| = |t| + 1$ is positive even. Let N be a semi-free DG B-module. Since N is a graded free $B^\natural$-module, there is a graded free $A^\natural$-module M satisfying $N^\natural = B^\natural \otimes_{A^\natural} M$ as graded $B^\natural$-modules.

In the rest of this article, we work in the setting of Notation 2.

Since $|X|$ is even, note that

$$B^\natural = \bigoplus_{i\geqq 0} X^{(i)}A^\natural \tag{3.1}$$

where the right hand side is a direct sum of right $A^\natural$-modules. From the decomposition (3.1), $N^\natural$ can be described as follows;

$$N^\natural = \bigoplus_{i\geqq 0} X^{(i)}M. \tag{3.2}$$

Now let r be an integer and let f be a graded R-linear homomorphism from $N^\natural$ to $N^\natural(r)$, that is, f is R-linear with $f(N_n) \subseteq N_{n+r}$ for all $n \in \mathbb{Z}$. Given such an f, we consider the restriction of f on M. Along the decomposition (3.2), one can decompose $f|_M$ into the following form:

$$f|_M = \sum_{i\geq 0} X^{(i)} f_i, \tag{3.3}$$

where each f_i is a graded R-linear homomorphism from M to $M(r - i|X|)$. For $m \in M$, there is a unique decomposition $f(m) = \sum_i X^{(i)}m_i$ with $m_i \in M$ along (3.2). Then f_i is defined by $f_i(m) = m_i$. Note that the decomposition (3.3) is unique as long as we work under the fixed setting (3.2). We call the equality (3.3) *the expansion of* $f|_M$ and often call f_0 *the constant term of* $f|_M$.

Taking the expansion of $f|_M$ as in (3.3), we consider a graded R-linear homomorphism

$$\frac{d}{dX} f|_M = \sum_{i\geq 0} X^{(i)} f_{i+1}.$$

Note that $\frac{d}{dX}f|_M$ is a mapping from M to $N(r-|X|)$. The mapping $\frac{d}{dX}f|_M$ can be extended to an R-linear mapping $j(f)$ on N by setting $j(f)(X^{(i)}m_i) = X^{(i)}\frac{d}{dX}f|_M(m_i)$ for each $i \geq 0$ and $m_i \in M$. Thus we have a graded R-linear homomorphism $j(f)$ from N to $N(r-|X|)$.

Summing up the argument above, we define the j-operator on $\mathrm{Hom}_R(N,N)$ as follows:

Definition 3. We work in the setting of Notation 2. Then one can define a graded R-linear mapping $j : \mathrm{Hom}_R(N,N) \to \mathrm{Hom}_R(N,N)(-|X|)$, which we call the *$j$-operator* on $\mathrm{Hom}_R(N,N)$.

Remark 4. The notion of j-operator was first introduced by J. Tate in the paper [8] and extensively used by T.H. Gulliksen and G. Levin [4].

We say that a graded R-linear mapping $\delta : N \to N(-1)$ is a *B-derivation* if it satisfies $\delta(bn) = d^B(b)n + (-1)^{|b||\delta|}b\delta(n)$ for $b \in B$ and $n \in N$. Then $\mathrm{Der}_B(N)$ denotes the set of all B-derivations on N. Recall that $\mathrm{Hom}_B(N,N)$ is a set of the all $B^\natural$-linear endomorphisms on N. We note that both $\mathrm{Hom}_B(N,N)$ and $\mathrm{Der}_B(N)$ are subsets of $\mathrm{Hom}_R(N,N)$.

Lemma 5. *The following assertions hold for $f, g \in \mathrm{Hom}_B(N,N)$ and $\delta, \delta' \in \mathrm{Der}_B(N)$.*

(1) *$f = g$ if and only if $f|_M = g|_M$.*
(2) *$\delta = \delta'$ if and only if $\delta|_M = \delta'|_M$.*

We summarize some properties of the j-operator.

Lemma 6. *The following assertions hold.*

(1) *If f is in $\mathrm{Hom}_B(N,N)$, then so is $j(f)$.*
(2) *If δ is in $\mathrm{Der}_B(N)$, then $j(\delta)$ is in $\mathrm{Hom}_B(N,N)$ and the constant term δ_o of the expansion of $\delta|_M$ is an A-derivation on M.*

Lemma 7. *The following equalities hold for $f, g \in \mathrm{Hom}_B(N,N)$ and $\delta, \delta' \in \mathrm{Der}_B(N)$.*

(1) *$j(fg) = j(f)g + fj(g)$.*
(2) *$j(\delta\delta')|_M = j(\delta)\delta'|_M + \delta j(\delta')|_M$.*
(3) *If f is invertible, then $j(f\delta f^{-1}) = j(f)\delta f^{-1} + fj(\delta)f^{-1} + f\delta j(f^{-1})$.*

The differential mapping ∂^N on N is a B-derivation. From Lemma 7, we see that $j(\partial^N)$ is $B^\natural$-linear. This specific element of $\mathrm{Hom}_B(N,N)$ will be a key object when we consider the lifting property of N in the following argument. This is the reason why we make the following definition of Δ_N as

$$\Delta_N := j(\partial^N). \tag{3.4}$$

Recall again from Lemma 7 that Δ_N is a $B^\natural$-linear endomorphism on N such that $|\Delta_N| = -|X| - 1$ is an odd integer.

Remark 8. The exactly same definition was made by S. Nasseh and Y. Yoshino in the case where $|X|$ is odd. See [6, Convention 2.5].

Lemma 9. *It holds that*

$$\Delta_N \partial^N = -\partial^N \Delta_N.$$

Hence Δ_N is a cycle of degree $-|X|-1$ in $\mathrm{Hom}_B(N,N)$.

Proof. By using Proposition 7(2), we see that $0 = j(\partial^N\partial^N)|_M = j(\partial^N)\partial^N|_M + \partial^N j(\partial^N)|_M$. The mapping $j(\partial^N)$ is B-linear from Lemma 6. It is easily seen that $j(\partial^N)\partial^N + \partial^N j(\partial^N)$ is also B-linear. Hence we conclude that $j(\partial^N)\partial^N + \partial^N j(\partial^N) = 0$ from Lemma 5(1). By definition, $\Delta_N = j(\partial^N)$ is a cycle of degree $-|X| - 1$ in $\mathrm{Hom}_B(N, N)$. ☐

The following is basic and crucial for our lifting problem.

Lemma 10. *In the setting of Notation* 2*, the following assertions are equivalent:*

(1) $\Delta_N = 0$ *as an element of* $\mathrm{Hom}_B(N, N)$.

(2) *The graded A-module M has structure of a DG A-module and $N = B \otimes_A M$ holds as an equality of DG B-modules.*

Proof. We show only the implication (1) ⇒ (2). In the expansion $\partial^N|_M = \bigoplus_{i \geq 0} X^{(i)}\alpha_i$, that $\Delta_N = 0$ implies that $\alpha_i = 0$ for $i > 0$. Therefore $\partial^N|_M = \alpha_0$ is an A-derivation on M and (M, α_0) defines a DG A-module. Moreover we have $\partial^N = B \otimes_A \alpha_0$. Thus $N = B \otimes_A M$ as DG B-modules. Similarly, one can prove the converse (2) ⇒ (1). ☐

We denote by $[\Delta_N]$ a cohomology class in $\mathrm{Ext}_B^{|X|+1}(N, N) = H_{-|X|-1}(\mathrm{Hom}_B(N, N))$ which is defined by Δ_N from Lemma 9. As we show in the following main theorem the class $[\Delta_N]$ gives a precise obstruction for N to be liftable to A.

Theorem 11. *We work in the setting of Notation* 2*. We consider the following conditions:*

(1) *N is liftable to A.*

(2) $[\Delta_N] = 0$ *in* $\mathrm{Ext}_B^{|X|+1}(N, N)$.

(3) *N is weakly liftable to A.*

Then the implications (1) ⇒ (2) ⇐ (3) *hold. If N is bounded below as a graded module, then the implications* (1) ⇐ (2) ⇒ (3) *hold true.*

We omit the proof of Theorem 11. See [7] for details. The next proposition is a key to prove the implication (2) ⇒ (1) in this theorem.

Proposition 12. *We work in the setting of Notation* 2*. Let f be in* $\mathrm{Hom}_B(N, N)$ *of degree $-|X|$ and h be in* $\mathrm{Hom}_A(M, M)$ *of degree* 0*. Then there is a graded $B^\natural$-linear endomorphism g of degree* 0 *on N satisfying the following conditions:*

(1) $j(g) = gf$.

(2) *The constant term of the expansion of $g|_M$ is h.*

We showed the uniqueness of liftings.

Theorem 13. *We work in the setting of Notation* 2*. If N is liftable to A and* $\mathrm{Ext}_B^{|X|}(N, N) = 0$*, then a lifting of N is unique up to DG isomorphisms over A.*

Finally, we pose an open question.

Question 14. Let A be a DG R-algebra. We denote by $B = A\langle X_1, \cdots, X_n | dX_1 = t_1, \cdots, dX_n = t_n\rangle$ an extended DG R-algebra obtained by repeated the adjunction of free variables $X_1, \cdots, X_n$. Let N be a semi-free DG B-module. If $\mathrm{Ext}_B^i(N, N) = 0$ for $i > 0$, then does it hold that N is weakly liftable to A?

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