Generalizations of the correspondence between quasi-hereditary algebras and directed bocses

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Setting

K : algebraically closed field
A : finite dimensional K-algebra with n simple modules
$A - \mathrm{mod}$: category of finitely generated left A-modules
$P_1, \ldots, P_n$: pairwise non-isomorphic indecomposable projective A-modules
$\Lambda = (\{1, \ldots, n\}, \leq)$: totally ordered set
$\Delta_1, \ldots, \Delta_n$: standard A-modules for Λ

$$\mathrm{Hom}_A(\Delta_i, \Delta_j) = 0 \text{ for } i > j,\ \mathrm{Ext}^1_A(\Delta_i, \Delta_j) = 0 \text{ for } i \geq j$$

$\overline{\Delta}_1, \ldots, \overline{\Delta}_n$: properly standard A-modules for Λ

$$\mathrm{End}_A(\overline{\Delta}_i) \cong K,\ \mathrm{Hom}_A(\overline{\Delta}_i, \overline{\Delta}_j) = 0 \text{ for } i > j,\ \mathrm{Ext}^1_A(\overline{\Delta}_i, \overline{\Delta}_j) = 0 \text{ for } i > j$$

$\mathcal{B} = (B, W)$: <u>bocs</u> with projective kernel $\overline{W} = \mathrm{Ker}\,\varepsilon$ (ε : counit of W)
i.e. B is a finite dimensional basic K-algebra and W is a B-coalgebra.

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In this talk, we generalize the relationship between quasi-hereditary algebras and directed bocses given by Koenig, Külshammer and Ovsienko. They showed the following result.

KKO theory [Koenig-Külshammer-Ovsienko '14]

We have a bijection

$$\{\text{Morita equivalent classes of quasi-hereditary algebras}\}$$
$$\updownarrow$$
$$\{\text{Equivalence classes of the module categories over directed bocses}\}\,.$$

Let a quasi-hereditary algebra A and a directed bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra.

This includes the result by Brzeziński, Koenig and Külshammer.

Aim

We have a bijection

$$\{\text{Morita equivalent classes of } \Delta\text{-filtered (resp. } \overline{\Delta}\text{-filtered) algebras}\}$$
$$\updownarrow$$
$$\left\{\begin{matrix}\text{Equivalence classes of the module categories over} \\ \text{weakly directed (resp. one-cyclic directed) bocses}\end{matrix}\right\}.$$

Let a Δ-filtered (resp. $\overline{\Delta}$-filtered) algebra A and a weakly directed (resp. one-cyclic directed) bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact (resp. proper) Borel subalgebra.

Definitions

Algebras (QHA,Δ-FA, $\overline{\Delta}$-FA)

1. An algebra A is Δ-filtered (or a Δ-FA) if there is some Λ such that ${}_A A$ is filtered by Δ. Similarly, a $\overline{\Delta}$-FA is defined.
2. An algebra A is quasi-hereditary (or a QHA) if it is a Δ-FA and $\Delta = \overline{\Delta}$ for Λ.

Hereafter, assume that every totally ordered set Λ is natural, i.e., $\{1 < 2 < \cdots < n\}$.

Bocses (DB,WDB,ODB)

Let $e_1, \ldots, e_n$ be pairwise orthogonal primitive idemp. of B. Then a bocs $\mathcal{B} = (B, W)$ is called

$$\begin{cases} \text{directed (DB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i \leq j \text{ and } \overline{W} \cong \bigoplus_{i>j} (Be_i \otimes_K e_j B)^{d_{ij}} \\ \text{weakly directed (WDB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i \leq j \text{ and } \overline{W} \cong \bigoplus_{i\geq j} (Be_i \otimes_K e_j B)^{d_{ij}} \\ \text{one-cyclic directed (ODB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i < j \text{ and } \overline{W} \cong \bigoplus_{i>j} (Be_i \otimes_K e_j B)^{d_{ij}} \end{cases}$$

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KKO theory was shown by three lemmas.

(1) [Dlab-Ringel '90]

Let $\mathcal{C}$ be an abelian category with standardizable set $\Theta = \{\Theta_1, \dots, \Theta_n\}$ i.e.

$$\mathrm{End}(\Theta_i) \cong K,\ \mathrm{Hom}(\Theta_i, \Theta_j) = 0 \text{ for } i > j, \text{ and } \mathrm{Ext}^1(\Theta_i, \Theta_j) = 0 \text{ for } i \geq j.$$

Then there is a unique QHA A, up to Morita equivalence, such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

(2) [Burt-Butler '91], [Brzeziński-Koenig-Külshammer '20], [KKO]

Let $\mathcal{B} = (B, W)$ be a DB. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is a QHA such that $\mathcal{B} - \mathrm{mod} \simeq \mathcal{F}(\Delta_{R_{\mathcal{B}}})$, and B is a homological exact Borel subalgebra of $R_{\mathcal{B}}$.

(3) [KKO]

Let A be a QHA. Then there exists a DB $\mathcal{B}_A$ such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A - \mathrm{mod}$.

$$\begin{tikzcd}
\text{QHA} & A \arrow[rr, "\text{(3)[KKO]}"] \arrow[dr, leftrightarrow, "\text{(1)[DR]}"'] & & \mathcal{B} \arrow[dl, "\text{(2)[BB]}"] & \text{DB} \\
& & R & \text{QHA} &
\end{tikzcd}$$

(1) [Dlab-Ringel '90]

Let $\mathcal{C}$ be an abelian category with standardizable set $\Theta = \{\Theta_1, \dots, \Theta_n\}$ i.e.

$$\mathrm{End}(\Theta_i) \cong K, \ \mathrm{Hom}(\Theta_i, \Theta_j) = 0 \text{ for } i > j, \text{ and } \mathrm{Ext}^1(\Theta_i, \Theta_j) = 0 \text{ for } i \geq j.$$

Then there is a unique QHA A, up to Morita equivalence, such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

Sketch of the proof of (1)

1. Put $P(i,i) = \Theta_i$ and inductively define $P(i,m)$ as a universal extension of Θ_m by $P(i,m-1)$.
2. We get $P(i,n)$, which is an Ext-projective object in $\mathcal{F}(\Theta)$.
3. Let $A = \mathrm{End}_{\mathcal{C}}(\bigoplus_{i=1}^n P(i,n))$. Then A is a QHA such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

(2) [Burt-Butler '91], [Brzeziński-Koenig-Külshammer '20], [KKO]

Let $\mathcal{B} = (B, W)$ be a DB. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is a QHA such that $\mathcal{B} - \mathrm{mod} \simeq \mathcal{F}(\Delta_{R_{\mathcal{B}}})$, and B is a homological exact Borel subalgebra of $R_{\mathcal{B}}$.

Sketch of the proof of (2)

1. We define $R = R_{\mathcal{B}} = \mathrm{Hom}_{B^{\mathrm{op}} \otimes B}(W, \mathrm{End}_K(B))$ and put $\Delta_R(i) = R \otimes_B S_B(i)$ where $S_B(i)$ are simple B-modules.
2. Then $\mathcal{B} - \mathrm{mod} \simeq \mathrm{Ind}(B, R) \simeq \mathcal{F}(\Delta_R)$.
3. Moreover Δ_R is a standardizable set in $R - \mathrm{mod}$ since the maps

$$\mathrm{Ext}^1_B(X, Y) \to \mathrm{Ext}^1_R(R \otimes X, R \otimes Y),\ \ X, Y : B\text{-modules}$$

are surjective.

(3) [KKO]

Let A be a QHA. Then there exists a DB $\mathcal{B}_A$ such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A - \mathrm{mod}$.

Sketch of the proof of (3)

1. Let $\mathcal{A} = \mathrm{Ext}_A^*(\Delta, \Delta)$ and $Q = Ds\mathcal{A}$ where s is a suspension for a graded algebra $\mathcal{A}$.
2. $U = T[Q]/I$ where $T[Q]$ is a tensor algebra of Q over $\bigoplus K\,\mathrm{id}_{\Delta_i}$ and I is an ideal generated by $Q^{\leq -1}$ and $d(Q^{-1})$. This d is a differential of $T[Q]$ and induced from graded maps for an A_∞-algebra $\mathcal{A}$.
3. $B = T[Q^0]/T[Q^0] \cap I$ and $W = U_1/d(B)$ form a DB $\mathcal{B} = (B, W)$ such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A - \mathrm{mod}$.

Example 1-1

$$\begin{tikzcd} A = K(1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_2"])/ < \alpha_2\alpha_1, \beta_1\beta_2, \alpha_1\beta_1 - \beta_2\alpha_2, \alpha_2\beta_2 >: \text{ QHA} \end{tikzcd}$$

$$P_1 = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix},\ P_2 = \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix},\ P_3 = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$$

$$\Delta_1 = \begin{smallmatrix} 1 \end{smallmatrix},\ \Delta_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix},\ \Delta_3 = \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$$

$$\dim \mathrm{rad}(\Delta_i, \Delta_{i+1}) = 1,\ \dim \mathrm{Ext}^1(\Delta_i, \Delta_{i+1}) = 1 \text{ for } i = 1, 2$$

$$\begin{tikzcd} \mathcal{B}:\ 1 \arrow[r, shift left, "a"] \arrow[r, shift right, dotted, "x"'] & 2 \arrow[r, shift left, "b"] \arrow[r, shift right, dotted, "y"'] & 3 \text{ with } ba = 0 : \text{ DB} \end{tikzcd}$$

This relation comes from a non-zero element $m_2(b, a) \in \mathrm{Ext}^2(\Delta_1, \Delta_3)$ where m_2 is the Yoneda product.

Example 1-2

$$\begin{tikzcd} \mathcal{B} : 1 \arrow[r, "a"] \arrow[r, dotted, "x"', shift right=2] & 2 \arrow[r, "b"] \arrow[r, dotted, "y"', shift right=2] & 3 \end{tikzcd} \text{ with } ba = 0 \Rightarrow$$

$$\begin{tikzcd} B = & K \arrow[r] \arrow[d, dotted] \arrow[dr, dotted] & K^2 \arrow[r] \arrow[d, dotted] \arrow[dr, dotted] & K^2 \arrow[d, dotted] \\ R : & & & \\ B = & K \arrow[r] & K^2 \arrow[r] & K^2 \end{tikzcd}$$

R is isomorphic to

$$\left\{ \left[\begin{matrix} x_3 & x_4 & y_2 & y_3 \\ & x_5 & & y_4 \\ & & x_1 & x_2 \\ & & & x_3 \end{matrix}\right] \middle| x_i, y_j \in K \right\} \cong K \left(\begin{tikzcd} & 2 \arrow[dl, "\beta_1"'] \arrow[dr, "\alpha_2"] & \\ 1 \arrow[dr, "\alpha_1"'] & & 3 \arrow[dl, "\beta_2"] \\ & 2' & \end{tikzcd} \right) / < e_2 - e_{2'}, \beta_1\alpha_1 - \alpha_2\beta_2 >$$

$$\cong K\left(\begin{tikzcd} 1 \arrow[r, shift left, "\alpha_1"] & 2 \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & 3 \arrow[l, shift left, "\beta_2"] \end{tikzcd} \right) / \left\langle \begin{matrix} \alpha_2\alpha_1, \beta_1\beta_2, \\ \alpha_1\beta_1 - \beta_2\alpha_2, \alpha_2\beta_2 \end{matrix} \right\rangle .$$

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$$\begin{tikzcd}
\text{QHA} & A \arrow[rr, "(3)\text{[KKO]}"] \arrow[dr, leftrightarrow, "(1)\text{[DR]}"'] & & \mathcal{B} \arrow[dl, "(2)\text{[BB]}"] & \text{DB} \\
& & R & \text{QHA} &
\end{tikzcd}$$

QHA

Δ-FA

$\overline{\Delta}$-FA

$$\begin{tikzcd}
A \arrow[rr, "\text{(3)[KKO]}", "\text{Lemma (3)'}"'] \arrow[dr, leftrightarrow, "\text{[ADL]}", "\text{(1)[DR]}"'] & & \mathcal{B} \arrow[dl, "\text{(2)[BB]}"] \\
& R &
\end{tikzcd}$$

DB

WDB

ODB

QHA

Δ-FA

$\overline{\Delta}$-FA

The conditions $\mathrm{End}_A(\Delta_i) \cong K$ are not essential for KKO theory. Hence we conclude that the following theorem.

Δ-filtered algebras v.s. weakly directed bocses

We have a bijection

$$\{\text{Morita equivalent classes of } \Delta\text{-filtered algebras}\}$$
$$\updownarrow$$
$$\{\text{Equivalence classes of the module categories over weakly directed bocses}\}.$$

Let a Δ-filtered algebra A and a weakly directed bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra.

Remark : This is induced from the result by Bautista, Pérez and Salmerón in 2020.

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When we apply KKO theory to $\overline{\Delta}$-filtered algebras, we must consider the following two problems.

Problem 1

The first problem corresponds to Lemma (1). For a set Θ satisfying

$$\mathrm{End}(\Theta_i) \cong K, \ \mathrm{Hom}(\Theta_i, \Theta_j) = 0 \text{ for } i > j, \text{ and } \mathrm{Ext}^1(\Theta_i, \Theta_j) = 0 \text{ for } i < j,$$

we can not guarantee that $P(i, n)$ can be constructed within finitely many steps. So we use the next lemma.

Ágoston-Dlab-Lukács '08

Let A' be a finite dimensional algebra. Then there exists a unique $\overline{\Delta}$-FA A, up to Morita equivalence, such that $\mathcal{F}(\overline{\Delta}_A) \simeq \mathcal{F}(\overline{\Delta}_{A'})$.

Problem 2

The second problem corresponds to Lemma (3).

$$A: \text{ QHA} \Rightarrow \mathrm{gldim}\ A < \infty \Rightarrow \dim \mathrm{Ext}_A^*(\Delta, \Delta) < \infty$$

But in general, for a $\overline{\Delta}$-FA A, the algebra $\mathcal{A} = \mathrm{Ext}_A^*(\overline{\Delta}, \overline{\Delta})$ is not finite dimensional. Consider a factor algebra $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{A}^{\geq 3}$ of $\mathcal{A}$, which finite dimensional.

Lemma (3)'

Let A be a $\overline{\Delta}$-FA and $\mathcal{A} = \mathrm{Ext}_A^*(\overline{\Delta}, \overline{\Delta})$. We add the step

⓪ Let $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{A}^{\geq 3}$.

before ① in the sketch of the proof and change $\mathcal{A}$ to $\overline{\mathcal{A}}$ in the steps thereafter. Then we get an ODB $\overline{\mathcal{B}_A}$ such that $\mathcal{F}(\overline{\Delta}_A) \simeq \overline{\mathcal{B}_A} - \mathrm{mod}$.

When A is a QHA, $\overline{\mathcal{B}_A} = \mathcal{B}_A$.

As a consequence, we get the following.

$\overline{\Delta}$-filtered algebras v.s. one-cyclic directed bocses

We have a bijection

$$\{\text{Morita equivalent classes of } \overline{\Delta}\text{-filtered algebras}\}$$
$$\updownarrow$$
$$\{\text{Equivalence classes of the module categories over one-cyclic directed bocses}\}.$$

Let a $\overline{\Delta}$-filtered algebra A and a one-cyclic directed bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological proper Borel subalgebra.

Example 2

$$\begin{tikzcd} A = K(1 \arrow[loop above, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[r, "\gamma"] & 3)/ < \alpha^2, \gamma\beta\alpha >: \ \overline{\Delta}\text{-FA} \end{tikzcd}$$

$$P_1 = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ 2 & & 3 \end{smallmatrix}, \ P_2 = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \ P_3 = \begin{smallmatrix} 3 \end{smallmatrix},$$

The properly standard modules are simple.

$$\dim \operatorname{Ext}^k(\overline{\Delta}_1, \overline{\Delta}_1) = 1 \text{ for } k \geq 0, \ \dim \operatorname{Ext}^1(\overline{\Delta}_i, \overline{\Delta}_{i+1}) = 1 \text{ for } i = 1, 2$$

$$\begin{tikzcd} \overline{\mathcal{B}_A} : \ 1 \arrow[loop above, "a"] \arrow[r, "b"] & 2 \arrow[r, "c"] & 3 \ \text{ with } a^2 = 0, cba = 0 : \ \text{ODB} \end{tikzcd}$$

These relations come from non-zero elements $m_2(a, a) \in \operatorname{Ext}^2(\overline{\Delta}_1, \overline{\Delta}_1)$, $m_3(c, b, a) \in \operatorname{Ext}^2(\overline{\Delta}_1, \overline{\Delta}_3)$, where $m_3 : \overline{\mathcal{A}}^{\otimes 3} \to \overline{\mathcal{A}}$ is a map of degree -1 for an A_∞-algebra $\overline{\mathcal{A}}$.

Thank you for your attention.