Generalizations of the correspondence between quasi-hereditary algebras and directed bocses

Yuichiro GOTO

2021/09/06



1 Background

2 KKO theory

3 Generalization for Δ -filtered algebras

Setting

- K: algebraically closed field
- A: finite dimensional K-algebra with n simple modules
- A mod: category of finitely generated left A-modules
- P_1, \ldots, P_n : pairwise non-isomorphic indecomposable projective A-modules
- $\Lambda = (\{1,\ldots,n\},\leq)$: totally ordered set
- Δ_1,\ldots,Δ_n : standard A-modules for Λ

$$\operatorname{Hom}_A(\Delta_i, \Delta_j) = 0 \text{ for } i > j, \ \operatorname{Ext}_A^1(\Delta_i, \Delta_j) = 0 \text{ for } i \ge j$$

 $\overline{\Delta}_1, \ldots, \overline{\Delta}_n$: properly standard A-modules for Λ

 $\operatorname{End}_A(\overline{\Delta}_i) \cong K, \ \operatorname{Hom}_A(\overline{\Delta}_i, \overline{\Delta}_j) = 0 \text{ for } i > j, \ \operatorname{Ext}^1_A(\overline{\Delta}_i, \overline{\Delta}_j) = 0 \text{ for } i > j$

 $\mathcal{B} = (B, W)$: <u>bocs</u> with projective kernel $\overline{W} = \operatorname{Ker} \varepsilon$ (ε : counit of W) i.e. B is a finite dimensional basic K-algebra and W is a B-coalgebra.

Contents

1 Background

2 KKO theory

3 Generalization for Δ -filtered algebras

In this talk, we generalize the relationship between quasi-hereditary algebras and directed bocses given by Koenig, Külshammer and Ovsienko. They showed the following result.

KKO theory [Koenig-Külshammer-Ovsienko '14]

We have a bijection

{Morita equivalent classes of quasi-hereditary algebras}

{Equivalence classes of the module categories over directed bocses}.

Let a quasi-hereditary algebra A and a directed bocs \mathcal{B} correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra.

This includes the result by Brzeziński, Koenig and Külshammer.

We have a bijection

Let a Δ -filtered (resp. $\overline{\Delta}$ -filtered) algebra A and a weakly directed (resp. one-cyclic directed) bocs \mathcal{B} correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact (resp. proper) Borel subalgebra.

Definitions

Algebras (QHA, Δ -FA, $\overline{\Delta}$ -FA)

1 An algebra A is Δ -filtered (or a Δ -FA) if there is some Λ such that $_AA$ is filtered by Δ . Similarly, a $\overline{\Delta}$ -FA is defined.

2 An algebra A is quasi-hereditary (or a QHA) if it is a Δ -FA and $\Delta = \overline{\Delta}$ for Λ .

Hereafter, assume that every totally ordered set Λ is natural, i.e., $\{1 < 2 < \cdots < n\}$.

Bocses (DB,WDB,ODB)

Let e_1, \ldots, e_n be pairwise orthogonal primitive idemp. of B. Then a bocs $\mathcal{B} = (B, W)$ is called

 $\begin{cases} \text{directed (DB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i \leq j \text{ and } \overline{W} \cong \bigoplus_{i > j} (Be_i \otimes_K e_j B)^{d_{ij}} \\ \text{weakly directed (WDB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i \leq j \text{ and } \overline{W} \cong \bigoplus_{i \geq j} (Be_i \otimes_K e_j B)^{d_{ij}} \\ \text{one-cyclic directed (ODB)} & \text{if } \operatorname{rad}_B(Be_i, Be_j) = 0 \text{ for } i < j \text{ and } \overline{W} \cong \bigoplus_{i \geq j} (Be_i \otimes_K e_j B)^{d_{ij}} \end{cases}$

Contents

1 Background

2 KKO theory

3 Generalization for Δ -filtered algebras

KKO theory was shown by three lemmas.

(1) [Dlab-Ringel '90]

Let $\mathcal C$ be an abelian category with standardizable set $\Theta = \{\Theta_1, \dots, \Theta_n\}$ i.e.

 $\operatorname{End}(\Theta_i) \cong K$, $\operatorname{Hom}(\Theta_i, \Theta_j) = 0$ for i > j, and $\operatorname{Ext}^1(\Theta_i, \Theta_j) = 0$ for $i \ge j$.

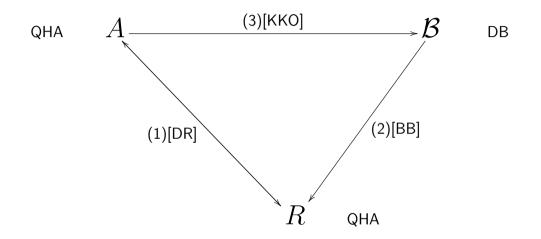
Then there is a unique QHA A, up to Morita equivalence, such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

(2) [Burt-Butler '91], [Brzeziński-Koenig-Külshammer '20], [KKO]

Let $\mathcal{B} = (B, W)$ be a DB. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is a QHA such that $\mathcal{B} - \text{mod} \simeq \mathcal{F}(\Delta_{R_{\mathcal{B}}})$, and B is a homological exact Borel subalgebra of $R_{\mathcal{B}}$.

(3) [KKO]

Let A be a QHA. Then there exists a DB \mathcal{B}_A such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A - \text{mod.}$



(1) [Dlab-Ringel '90]

Let C be an abelian category with standardizable set $\Theta = \{\Theta_1, \ldots, \Theta_n\}$ i.e.

 $\operatorname{End}(\Theta_i) \cong K$, $\operatorname{Hom}(\Theta_i, \Theta_j) = 0$ for i > j, and $\operatorname{Ext}^1(\Theta_i, \Theta_j) = 0$ for $i \ge j$.

Then there is a unique QHA A, up to Morita equivalence, such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

Sketch of the proof of (1)

1 Put $P(i,i) = \Theta_i$ and inductively define P(i,m) as a universal extension of Θ_m by P(i,m-1).

2 We get P(i, n), which is an Ext-projective object in $\mathcal{F}(\Theta)$.

3 Let $A = \operatorname{End}_{\mathcal{C}}(\bigoplus_{i=1}^{n} P(i, n))$. Then A is a QHA such that $\mathcal{F}(\Theta) \simeq \mathcal{F}(\Delta_A)$.

(2) [Burt-Butler '91], [Brzeziński-Koenig-Külshammer '20], [KKO]

Let $\mathcal{B} = (B, W)$ be a DB. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is a QHA such that $\mathcal{B} - \text{mod} \simeq \mathcal{F}(\Delta_{R_{\mathcal{B}}})$, and B is a homological exact Borel subalgebra of $R_{\mathcal{B}}$.

Sketch of the proof of (2)

- **1** We define $R = R_{\mathcal{B}} = \operatorname{Hom}_{B^{\operatorname{op}} \otimes B}(W, \operatorname{End}_{K}(B))$ and put $\Delta_{R}(i) = R \otimes_{B} S_{B}(i)$ where $S_{B}(i)$ are simple *B*-modules.
- **2** Then $\mathcal{B} \text{mod} \simeq \text{Ind}(B, R) \simeq \mathcal{F}(\Delta_R)$.

(3) Moreover Δ_R is a standardizable set in R - mod since the maps $\operatorname{Ext}^1_B(X, Y) \to \operatorname{Ext}^1_R(R \otimes X, R \otimes Y), \ X, Y : B$ -modules

are surjective.

(3) [KKO]

Let A be a QHA. Then there exists a DB \mathcal{B}_A such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A - \text{mod.}$

Sketch of the proof of (3)

- 1 Let $\mathcal{A} = \operatorname{Ext}_{\mathcal{A}}^*(\Delta, \Delta)$ and $Q = Ds\mathcal{A}$ where s is a suspension for a graded algebra \mathcal{A} .
- 2 U = T[Q]/I where T[Q] is a tensor algebra of Q over $\bigoplus K \operatorname{id}_{\Delta_i}$ and I is an ideal generated by $Q^{\leq -1}$ and $d(Q^{-1})$. This d is a differential of T[Q] and induced from graded maps for an A_{∞} -algebra \mathcal{A} .
- **3** $B = T[Q^0]/T[Q^0] \cap I$ and $W = U_1/d(B)$ form a DB $\mathcal{B} = (B, W)$ such that $\mathcal{F}(\Delta_A) \simeq \mathcal{B}_A \text{mod.}$

Example 1-1

$$A = K(1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3) / < \alpha_2 \alpha_1, \beta_1 \beta_2, \alpha_1 \beta_1 - \beta_2 \alpha_2, \alpha_2 \beta_2 >: \text{QHA}$$

$$P_1 = \frac{1}{2}, P_2 = 1 \frac{2}{2} 3, P_3 = \frac{3}{2}$$

$$\Delta_1 = 1, \Delta_2 = \frac{2}{1}, \Delta_3 = \frac{3}{2}$$

$$\dim \operatorname{rad}(\Delta_i, \Delta_{i+1}) = 1, \dim \operatorname{Ext}^1(\Delta_i, \Delta_{i+1}) = 1 \text{ for } i = 1, 2$$

$$\mathcal{B} : 1 \xrightarrow[x]{a} 2 \xrightarrow[y]{b} 3 \text{ with } ba = 0 : \text{ DB}$$

This relation comes from a non-zero element $m_2(b,a) \in \text{Ext}^2(\Delta_1, \Delta_3)$ where m_2 is the Yoneda product.

Example 1-2

$$\mathcal{B}: 1 \xrightarrow[x]{a} 2 \xrightarrow[y]{b} 3 \text{ with } ba = 0 \Rightarrow \begin{array}{c} B = \\ R: \\ B = \end{array} \xrightarrow[y]{k} K^2 \longrightarrow K^2 \\ R: \\ K \longrightarrow K^2 \longrightarrow K^2 \end{array}$$

 \boldsymbol{R} is isomorphic to

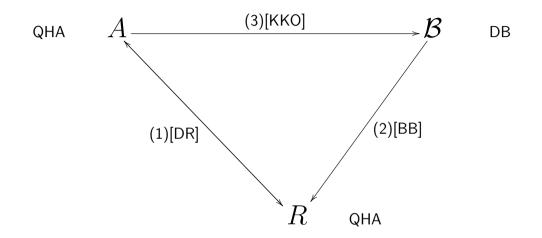
$$\begin{cases} \begin{bmatrix} x_3 & x_4 & y_2 & y_3 \\ x_5 & y_4 \\ & x_1 & x_2 \\ & & & x_3 \end{bmatrix} \middle| x_i, y_j \in K \end{cases} \cong K \begin{pmatrix} \beta_1 \swarrow^2 \swarrow^{\alpha_2} \\ 1 & 3 \\ \alpha_1 \swarrow^2 \swarrow' \beta_2 \end{pmatrix} / < e_2 - e_{2'}, \beta_1 \alpha_1 - \alpha_2 \beta_2 >$$
$$\cong K (1 \xrightarrow{\alpha_1}_{\overbrace{\beta_1}} 2 \xrightarrow{\alpha_2}_{\overbrace{\beta_2}} 3) / \begin{pmatrix} \alpha_2 \alpha_1, \beta_1 \beta_2, \\ \alpha_1 \beta_1 - \beta_2 \alpha_2, \alpha_2 \beta_2 \end{pmatrix}.$$

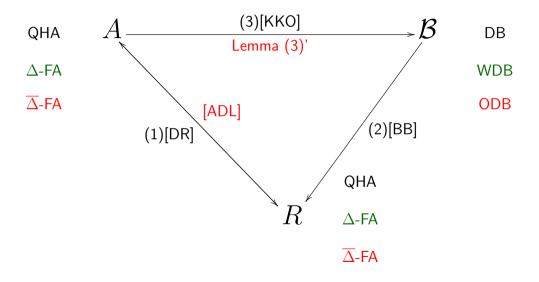
Contents

1 Background

2 KKO theory

3 Generalization for Δ -filtered algebras





The conditions $\operatorname{End}_A(\Delta_i) \cong K$ are not essential for KKO theory. Hence we conclude that the following theorem.

 Δ -filtered algebras v.s. weakly directed bocses

We have a bijection

```
{Morita equivalent classes of \Delta-filtered algebras}
```

\$

{Equivalence classes of the module categories over weakly directed bocses}.

Let a Δ -filtered algebra A and a weakly directed bocs \mathcal{B} correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra.

Remark : This is induced from the result by Bautista, Pérez and Salmerón in 2020.

Contents

1 Background

2 KKO theory

3 Generalization for Δ -filtered algebras

When we apply KKO theory to $\overline{\Delta}$ -filtered algebras, we must consider the following two problems.

Problem 1

The first problem corresponds to Lemma (1). For a set Θ satisfying

 $\operatorname{End}(\Theta_i) \cong K$, $\operatorname{Hom}(\Theta_i, \Theta_j) = 0$ for i > j, and $\operatorname{Ext}^1(\Theta_i, \Theta_j) = 0$ for i < j,

we can not guarantee that P(i, n) can be constructed within finitely many steps. So we use the next lemma.

Ágoston-Dlab-Lukács '08

Let A' be a finite dimensional algebra. Then there exists a unique $\overline{\Delta}$ -FA A, up to Morita equivalence, such that $\mathcal{F}(\overline{\Delta}_A) \simeq \mathcal{F}(\overline{\Delta}_{A'})$.

Problem 2

The second problem corresponds to Lemma (3).

$$A: \mathsf{QHA} \Rightarrow \mathrm{gldim} \ A < \infty \Rightarrow \mathrm{dim} \operatorname{Ext}^*_A(\Delta, \Delta) < \infty$$

But in general, for a $\overline{\Delta}$ -FA A, the algebra $\mathcal{A} = \operatorname{Ext}_{A}^{*}(\overline{\Delta}, \overline{\Delta})$ is not finite dimensional. Consider a factor algebra $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{A}^{\geq 3}$ of \mathcal{A} , which finite dimensional.

Lemma (3)'

Let A be a
$$\overline{\Delta}$$
-FA and $\mathcal{A} = \operatorname{Ext}_{A}^{*}(\overline{\Delta}, \overline{\Delta})$. We add the step

0 Let
$$\overline{\mathcal{A}} = \mathcal{A}/\mathcal{A}^{\geq 3}$$
.

before (1) in the sketch of the proof and change \mathcal{A} to $\overline{\mathcal{A}}$ in the steps thereafter. Then we get an ODB $\overline{\mathcal{B}}_A$ such that $\mathcal{F}(\overline{\Delta}_A) \simeq \overline{\mathcal{B}}_A - \text{mod.}$

When A is a QHA, $\overline{\mathcal{B}_A} = \mathcal{B}_A$.

As a consequence, we get the following.

 $\overline{\Delta}$ -filtered algebras v.s. one-cyclic directed bocses

We have a bijection

{Morita equivalent classes of $\overline{\Delta}$ -filtered algebras}

\$

{Equivalence classes of the module categories over one-cyclic directed bocses}.

Let a $\overline{\Delta}$ -filtered algebra A and a one-cyclic directed bocs \mathcal{B} correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of \mathcal{B} is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological proper Borel subalgebra. Example 2

$$A = K(\stackrel{\alpha}{1 \longrightarrow} 2 \stackrel{\gamma}{\longrightarrow} 3) / < \alpha^2, \gamma \beta \alpha >: \overline{\Delta} \text{-FA}$$
$$P_1 = \frac{1}{2} \stackrel{1}{_2} \frac{2}{_3}, \ P_2 = \frac{2}{_3}, \ P_3 = 3,$$

The properly standard modules are simple.

dim
$$\operatorname{Ext}^{k}(\overline{\Delta}_{1}, \overline{\Delta}_{1}) = 1$$
 for $k \ge 0$, dim $\operatorname{Ext}^{1}(\overline{\Delta}_{i}, \overline{\Delta}_{i+1}) = 1$ for $i = 1, 2$
 $\overline{\mathcal{B}_{A}} : \stackrel{\textcircled{a}}{1 \longrightarrow} 2 \stackrel{c}{\longrightarrow} 3$ with $a^{2} = 0, cba = 0$: ODB

These relations come from non-zero elements $m_2(a,a) \in \operatorname{Ext}^2(\overline{\Delta}_1,\overline{\Delta}_1), \ m_3(c,b,a) \in \operatorname{Ext}^2(\overline{\Delta}_1,\overline{\Delta}_3)$, where $m_3: \overline{\mathcal{A}}^{\otimes 3} \to \overline{\mathcal{A}}$ is a map of degree -1 for an A_{∞} -algebra $\overline{\mathcal{A}}$.

Thank you for your attention.