Characterization of the quantum projective planes finite over their centers

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Notation

- k: an algebraically closed field of characteristic 0.
- All graded algebras are finitely generated in degree 1 over k.
  - $k\langle x_1, \dots, x_n \rangle / I$
    ($\exists$ homog. ideal $I \lhd k\langle x_1, \dots, x_n \rangle$, $\deg x_i = 1$, $\forall i = 1, \dots, n$).
- $\operatorname{GrMod} A$: the cat. of graded right A-modules.
- $\operatorname{grmod} A$: the cat. of fin. gen. graded right A-modules.
- $\mathbb{P}_k^{n-1} (= \mathbb{P}^{n-1})$: the $n-1$-dim. proj. space over k.

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Quantum polynomial algebras

Definition 1.1 (Artin-Schelter, 1987)

A right noetherian graded algebra A is called a *d-dimensional quantum polynomial algebra* (*d-dim qpa*) if

(i) $\operatorname{gldim} A = d$,

(ii) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (*Gorenstein condition*)

(iii) $H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1 - t)^{-d}$ (*Hilbert series*).

Example

(1) $A := k[x_1, \ldots, x_d]$: comm. d-dim qpa.

(2) $A := k\langle x_1, \ldots, x_d\rangle/(x_j x_i - \alpha_{i,j} x_i x_j)$,
$(1 \leq i < j \leq d,\ \alpha_{i,j} \in k\backslash\{0\})$ (*Skew poly. alg.*) : d-dim qpa.

(3) A: 3-dim qpa $\Longleftrightarrow$ A: 3-dim. quad. AS-regular alg. ([AS]).
- $A \cong k\langle x, y, z\rangle/(f_1, f_2, f_3)$, $f_i \in k\langle x, y, z\rangle_2$ $(i = 1, 2, 3)$.

(4) $\forall d \geq 4$, A: d-dim qpa. (???)

Geometric algebras

- Geometric pair (E, σ): a proj. scheme $E \subset \mathbb{P}^{n-1}$, $\sigma \in \mathrm{Aut}_k E$.
- $A = k\langle x_1, \dots, x_n \rangle / I$ ($I \lhd k\langle x_1, \dots, x_n \rangle_2$) : quad. algebra,

$$\mathcal{V}(I_2) := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0, \ \forall f \in I_2\}.$$

Definition 1.2 (Mori, 2006)

A quad. algebra A is called *geometric* if $\exists (E, \sigma)$ such that

(G1) $\mathcal{V}(I_2) = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}$
(we write $\mathcal{P}(A) = (E, \sigma)$, E: the *point scheme* of A),

(G2) $I_2 = \{f \in k\langle x_1, \dots, x_n \rangle_2 \mid f(p, \sigma(p)) = 0, \ \forall p \in E\}$.
(we write $A = \mathcal{A}(E, \sigma)$).

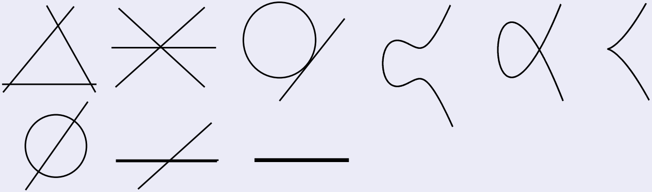
Example

E: a triangle in $\mathbb{P}^2$, σ stabilizes each component.
$\Longrightarrow A = \mathcal{A}(E, \sigma) = k\langle x, y, z \rangle / (yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$,
$\alpha, \beta, \gamma \in k \setminus \{0\}$, $\alpha\beta\gamma \neq 0, 1$: 3-dim geometric qpa.

ATV's theorem

Theorem 1.3 (Artin-Tate-Van den Bergh, 1990)

Every 3-dimensional quantum polynomial algebra is geometric where the point scheme is either $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$.



Remark 1.4

Note that the classification of 3-dim qpa $A = \mathcal{A}(E, \sigma)$ reduces to the classification of geometric pairs (E, σ).

Quantum projective spaces (quantum $\mathbb{P}^{d-1}$)

- A: a right noeth. graded algebra.
- $\operatorname{tors} A$: the full subcat. of $\operatorname{grmod} A$ consisting of fin. dim. modules over k.

Definition 2.1 (Artin-Zhang, 1994)

(1) *The noncommutative projective scheme associated to A* is defined by $\operatorname{Proj}_{\mathrm{nc}} A = (\operatorname{tails} A, \pi A)$ where
  - $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$ is the quot. cat.,
  - $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ is the quot. func., $A \in \operatorname{grmod} A$ is regular.

(2) A: d-dim qpa $\Longrightarrow \operatorname{Proj}_{\mathrm{nc}} A$ is called *a quantum $\mathbb{P}^{d-1}$*.
  - $d = 3 \Longrightarrow \operatorname{Proj}_{\mathrm{nc}} A$ is called *a quantum projective plane*.

Remark 2.2

- A: commutative $\Longrightarrow \operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj} A$.
- A: 2-dim qpa $\Longrightarrow \operatorname{Proj}_{\mathrm{nc}} A \cong \mathbb{P}^1$.

Relationship between 3-dim qpa A and $\mathrm{Proj}_{\mathrm{nc}} A$

Theorem 2.3 (Abdelgadir-Okawa-Ueda, 2014)

Let A and A' be 3-dim qpa.

$$\operatorname{grmod} A \cong \operatorname{grmod} A' \iff \mathrm{Proj}_{\mathrm{nc}} A \cong \mathrm{Proj}_{\mathrm{nc}} A'.$$

Lemma 2.4 (I.-Matsuno, 2021)

$\forall$ 3-dim qpa A, $\exists$ a 3-dim Calabi-Yau qpa A' such that $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ so that $\mathrm{Proj}_{\mathrm{nc}} A \cong \mathrm{Proj}_{\mathrm{nc}} A'$.

- A qpa A' is called *Calabi-Yau* if the Nakayama automorphism of A' is the identity.

Remark 2.5

Lemma 2.4 claims that every quantum projective plane has a 3-dim Calabi-Yau qpa as a homogeneous coordinate ring.

Characterization when 3-dim qpa is finite over its center

Theorem 2.6 (ATV, 1991)

$A = \mathcal{A}(E, \sigma)$: 3-dim qpa.

$$|\sigma| < \infty \iff A \text{ is finite over its center.}$$

- To prove Theorem 2.6, fat points of a quantum projective plane $\mathrm{Proj}_{\mathrm{nc}} A$ plays an essential role.
- By [Artin, 1992], if A is finite over its center and $E \neq \mathbb{P}^2$, then $\mathrm{Proj}_{\mathrm{nc}} A$ has a fat point, however, the converse is not true.

Definition 2.7

Let A be a graded algebra.

(1) *A point of* $\mathrm{Proj}_{\mathrm{nc}} A$ is an isom. class of a simple obj. of the form $\pi M \in \mathrm{tails}\, A$ where $M \in \mathrm{grmod}\, A$ such that $\lim_{i\to\infty} \dim_k M_i < \infty$.

(2) A point πM is called *fat* if $\lim_{i\to\infty} \dim_k M_i > 1$ (in this case, M is called *a fat point module over A*).

Norm $\|\sigma\|$

To check the existence of a fat point, the following was introduced.

Definition 2.8 (Mori, 2015)

For a geometric pair (E, σ) where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$,

$$\mathrm{Aut}_k(\mathbb{P}^{n-1}, E) := \{\phi|_E \in \mathrm{Aut}_k E \mid \phi \in \mathrm{Aut}_k \mathbb{P}^{n-1}\},$$

and $\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i \in \mathrm{Aut}_k(\mathbb{P}^{n-1}, E)\}$, which is called *the norm of σ*.

For a geometric pair (E, σ), clearly $\|\sigma\| \leq |\sigma|$.

Lemma 2.9 ((Mori, 2015), (Artin, 1992))

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa.

(1) $\|\sigma\| = 1 \iff E = \mathbb{P}^2$.

(2) $1 < \|\sigma\| < \infty \iff \mathrm{Proj}_{\mathrm{nc}} A$ *has a fat point.*

"$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center"/Aim

Definition 2.10 ((Mori, 2015), (I.-Mori))

Let A be a d-dim qpa. We say that $\mathsf{Proj}_{\mathrm{nc}} A$ is *finite over its center* if $\exists$ d-dim qpa A' finite over its center such that

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \ (\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A').$$

Theorem 2.11 (Mori, 2015)

$A = \mathcal{A}(E, \sigma)$: a 3-dim qpa where E is a triangle in $\mathbb{P}^2$, $\sigma \in \mathrm{Aut}_k E$.

$$\|\sigma\| < \infty \iff \mathsf{Proj}_{\mathrm{nc}} A \text{ is finite over its center.}$$

Aim

The aim of this research is to extend Theorem 2.11 to all types.

Main results

Theorem 1 (I.-Mori): Calabi-Yau case

If $A = \mathcal{A}(E, \sigma)$ is a 3-dim Calabi-Yau qpa, then $||\sigma|| = |\sigma^3|$, so TFAE.

(1) $|\sigma| < \infty$.
(2) $||\sigma|| < \infty$.
(3) A is finite over its center.
(4) $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center.

Main results

Definition 3.1 (Mori-Ueyama, 2013)

For a d-dim geometric qpa $A = \mathcal{A}(E, \sigma)$ with the Nakayama auto. $\nu \in \mathrm{Aut}\, A$, a new graded algebra $\overline{A} := \mathcal{A}(E, \nu^* \sigma^d)$ satisfying (G2).

Lemma 3.2 (Mori-Ueyama, 2013)

A, A': geometric qpa.

$$\operatorname{grmod} A \cong \operatorname{grmod} A' \iff \overline{A} \cong \overline{A'}.$$

Main Theorem (I.-Mori): general case

If $A = \mathcal{A}(E, \sigma)$ is a 3-dim qpa with the Nakayama auto. $\nu \in \mathrm{Aut} A$, then $||\sigma|| = |\nu^* \sigma^3|$, so TFAE.

(1) $|\nu^* \sigma^3| < \infty$.

(2) $||\sigma|| < \infty$.

(3) $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.

Corollary

By Main Theorem and Lemma 2.9, we have the following result.

Corollary 1 (I.-Mori)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa such that $E \neq \mathbb{P}^2$, and $\nu \in \mathrm{Aut} A$ the Nakayama auto. of A. Then TFAE.

(1) $|\nu^* \sigma^3| < \infty$.

(2) $\|\sigma\| < \infty$.

(3) $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.

(4) $\mathrm{Proj}_{\mathrm{nc}} A$ has a fat piont.

Example

Example

$A = \mathcal{A}(E,\sigma) = k\langle x,y,z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx)$, $0 \neq \alpha,\beta,\gamma \in k$, : 3-dim qpa, where $E = \mathcal{V}(x)\cup\mathcal{V}(y)\cup\mathcal{V}(z) \subset \mathbb{P}^2$,

$$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c), \\ \sigma(a,0,c) = (\beta a,0,c), \\ \sigma(a,b,0) = (a,\gamma b,0), \end{cases} \quad \nu^* = \begin{pmatrix} \gamma/\beta & 0 & 0 \\ 0 & \alpha/\gamma & 0 \\ 0 & 0 & \beta/\alpha \end{pmatrix},$$

$$\begin{cases} \nu^*\sigma^3(0,b,c) = (0,b,\alpha\beta\gamma c), \\ \nu^*\sigma^3(a,0,c) = (\alpha\beta\gamma a,0,c), \\ \nu^*\sigma^3(a,b,0) = (a,\alpha\beta\gamma b,0). \end{cases}$$

(1) $|\sigma| = \mathrm{lcm}(|\alpha|,|\beta|,|\gamma|) < \infty \iff A$ is finite over its center.

(2) $\|\sigma\| = |\nu^*\sigma^3| = |\alpha\beta\gamma| < \infty \iff \mathrm{Proj}_{\mathrm{nc}}A$ is finite over its center $\iff \mathrm{Proj}_{\mathrm{nc}}A$ has a fat piont.

Beilinson algebras of d-dim qpa

Definition 3.3 (Minamoto-Mori, 2011)

For a d-dim qpa A, *the Beilinson algebra of A* is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ 0 & A_0 & \cdots & A_{d-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

- The Beilinson algebra is a typical example of $(d-1)$-representation infinite algebra in the sense of [Herschend-Iyama-Oppermann, 2014] ([Minamoto-Mori, 2011]).
- To investigate representation theory of such an algebra, it is important to classify simple $(d-1)$-regular modules.

Applications

We finally apply our results to representation theory of finite dimensional algebras.

Corollary 2 (I.-Mori)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa with the Nakayama auto. $\nu \in \mathrm{Aut}\, A$. Then TFAE.

(1) $|\nu^* \sigma^3| = 1$ or ∞.

(2) $\mathsf{Proj}_{\mathrm{nc}} A$ has no fat point.

(3) The isomorphism classes of simple 2-regular modules over ∇A are parameterized by the set of closed points of $E \subset \mathbb{P}^2$.

Thank you for your attention !

If you have an interest in our talk, please see arXiv:2010.13093.

Proof of Theorem 1

- By calculation, $||\sigma|| = |\sigma^3|$ holds for each type. So, (1) $\Leftrightarrow$ (2).
- By Theorem 2.6, (1) $\Leftrightarrow$ (3). By definition, (3) $\Rightarrow$ (4).
- (4) $\Rightarrow$ (2): If $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center, then there exists a 3-dim qpa $A' = \mathcal{A}(E', \sigma')$ which is finite over its center such that $\mathrm{Proj}_{\mathrm{nc}} A \cong \mathrm{Proj}_{\mathrm{nc}} A'$ by Definition 2.10, so $\|\sigma\| = \|\sigma'\| \leq |\sigma'| < \infty$ by [Mori, 2015] and Theorem 2.6.

Proof of Main Theorem

- By Lemma 2.4, $\forall$ 3-dim qpa $A = \mathcal{A}(E, \sigma)$, $\exists$ a 3-dim Calabi-Yau qpa $A' = \mathcal{A}(E', \sigma')$ s. t. $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.
- Since the Nakayama auto. of A' is the identity, $\mathcal{A}(E, \nu^*\sigma^3) = \overline{A} \cong \overline{A'} = \mathcal{A}(E', {\sigma'}^3)$ by Lemma 3.2, so, by [Mori, 2015] and Theorem 1,

$$||\sigma|| = ||\sigma'|| = |{\sigma'}^3| = |\nu^*\sigma^3|.$$

$$\text{Proj}_{\text{nc}} A \text{ is fin. over its center} \overset{\text{[Mori, 2015]}}{\iff} \text{Proj}_{\text{nc}} A' \text{ is fin. over its center}$$

$$\overset{\text{Thm 1}}{\iff} ||\sigma'|| < \infty.$$

Therefore, we have the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).