Structure theorem for flat cotorsion modules over Noether algebras

arXiv: 2108.03153

by Ryo Kanda & Tsutomu Nakamura

(The 53rd Symposium on Ring Theory and Representation Theory in Japan
September 17, 2021.)

<u>Aim</u> Classify all flat cotorsion modules for Noether algebras in terms of prime ideals (generalizing [Enochs 1984] for comm noeth rings).

§1 Flat cotorsion modules

A : ring. $\operatorname{Mod} A := \{\text{right } A\text{-modules}\}$.

Def $M \in \mathrm{Mod}\,A$: flat

$:\Leftrightarrow M \otimes_A - : \mathrm{Mod}\,A^{op} \longrightarrow \mathrm{Mod}\,\mathbb{Z}$: exact.

$\mathrm{Flat}\,A := \{\text{flat modules}\} \subset \mathrm{Mod}\,A$.

$M \in \mathrm{Mod}\,A$: cotorsion

$:\Leftrightarrow \mathrm{Ext}^1_A(\mathrm{Flat}\,A, M) = 0$.

$\mathrm{Cot}\,A := \{\text{cotorsion modules}\} \subset \mathrm{Mod}\,A$.

$\mathrm{FlCot}\,A := \mathrm{Flat}\,A \cap \mathrm{Cot}\,A$.

Def $\mathcal{A}$: abelian cat. $\mathcal{X}, \mathcal{Y} \subset_{\text{full}} \mathcal{A}$.

$(\mathcal{X}, \mathcal{Y})$: cotorsion pair

$$:\Leftrightarrow \begin{cases} \mathcal{X} = \{ M \mid \mathrm{Ext}^1(M, \mathcal{Y}) = 0 \} \\ \mathcal{Y} = \{ M \mid \mathrm{Ext}^1(\mathcal{X}, M) = 0 \} \end{cases}$$

Moreover

hereditary $:\Leftrightarrow \mathrm{Ext}^{>0}(\mathcal{X}, \mathcal{Y}) = 0.$

complete $:\Leftrightarrow \forall M \in \mathcal{A},$

$$\begin{aligned} &0 \to Y \to X \to M \to 0 \\ &0 \to M \to Y' \to X' \to 0 \end{aligned} \quad \begin{pmatrix} X, X' \in \mathcal{X} \\ Y, Y' \in \mathcal{Y} \end{pmatrix}.$$

exact

<u>Ex</u> $(\mathrm{Proj} A, \mathrm{Mod} A)$, $(\mathrm{Mod} A, \mathrm{Inj} A)$.

Fact (Flat A, Cot A) is a complete hereditary cotorsion pair in Mod A.

Flat Cover Conjecture
solved by [Bican–El Bashir–Enochs 2001]

cf [Nakamura–Thompson 2020]

§2 Structure theorem.

R: comm noeth ring.

A: Noether R-algebra.

i.e. $\underset{\text{subring}}{R} \subset \underset{\text{center}}{Z(A)} \subset A$ & A is fin. gen. as an R-mod.

$$\begin{array}{ccc} \operatorname{Spec} A := \{\text{prime (two-sided) ideals of } A\}. & & \\ \text{1-1} \updownarrow \text{[Gabriel 1962]} & & \\ \{\text{indec injectives in } \operatorname{Mod} A\}/\cong & & \end{array}$$

$P \mapsto I_A(P)$

$E_A\left(\frac{A}{P}\right) \cong I_A(P)^{n_P}$

$$\begin{array}{ccc} \operatorname{Spec} A & \twoheadrightarrow & \operatorname{Spec} R \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ P & \longmapsto & P \cap R \end{array}$$

<u>Thm</u> [Kanda-Nakamura; Enochs when A=R]

$M \in \mathrm{Mod}\,A$; flat cotorsion

$$\Longleftrightarrow M \cong \prod_{p \in \mathrm{Spec}A} \mathrm{Hom}_R\left(I_{A^{op}}(p), E_R\left(\frac{R}{p \cap R}\right)^{\oplus B_p}\right)$$

set

(The cardinality of B_p is uniquely determined)

<u>Cor</u> $\{\text{indec flcot in Mod}A\}/\cong \overset{1-1}{\longleftrightarrow} \mathrm{Spec}\,A$

$$\begin{array}{ccc} \cup & & \cup \\ T_A(p) := \mathrm{Hom}_R\left(I_{A^{op}}(p), E_R\left(\frac{R}{p \cap R}\right)\right) & \longleftarrow & p \end{array}$$

($\exists$bij by [Herzog 1993])

Thm [KN]

$\forall \mathfrak{p} \in \operatorname{Spec} R$,

$\widehat{}$ ← $\mathfrak{p}$-adic completion

$$\widehat{A_{\mathfrak{p}}} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} T_A(P)^{\wedge \mathfrak{p}} \qquad \text{in } \operatorname{Mod} A$$

Ex $A = \begin{pmatrix} R & \\ R & R \end{pmatrix} = R(\cdot \longrightarrow \cdot)$.

$$\widehat{A_{\mathfrak{p}}} = \begin{pmatrix} \widehat{R_{\mathfrak{p}}} & \\ \widehat{R_{\mathfrak{p}}} & \widehat{R_{\mathfrak{p}}} \end{pmatrix} = \begin{matrix} (\widehat{R_{\mathfrak{p}}} \ \ 0) \\ \oplus \\ (\widehat{R_{\mathfrak{p}}} \ \ \widehat{R_{\mathfrak{p}}}) \end{matrix}.$$

$$\{\text{indec flat in } \operatorname{Mod} A\}/_{\cong} = \{(\widehat{R_{\mathfrak{p}}} \ \ 0), (\widehat{R_{\mathfrak{p}}} \ \ \widehat{R_{\mathfrak{p}}}) \mid \mathfrak{p} \in \operatorname{Spec} R\},$$

§3 Ziegler Spectrum.

$Zg_A := \{\text{indec pure-injectives in } \mathrm{Mod}\,A\}/_{\cong} \ni N$

$\curvearrowleft$ top sp.

$\{\text{indec flats}\}/_{\cong} \overset{\text{closed}}{\cup} \qquad \{\text{indec injs}\}/_{\cong} \overset{\text{closed}}{\cup}$

1-1 $\updownarrow$

$\{\text{indec injectives in } \mathrm{Func}^{\mathbb{Z}}(\mathrm{mod}\,A^{op}, \mathrm{Mod}\,\mathbb{Z})\}/_{\cong} \ni N \otimes_A -$

<u>Fact</u> (Elementary duality)

$$\{\text{open subsets of } Zg_A\} \underset{\text{poset}}{\cong} \{\text{open subsets of } Zg_{A^{op}}\}.$$

<u>Thm</u>

$$\begin{array}{ccc} Zg_A & & Zg_{A^{op}} \\ \cup & & \cup \\ \mathrm{flat}_A & \xrightarrow{\exists!} & \mathrm{inj}_{A^{op}} \end{array}$$

that is compatible w. elementary duality.

[Herzog].

This is given by $T_A(P) \mapsto I_{A\varphi}(P)$ [KN].