LOCALIZATION OF EXTRIANGULATED CATEGORIES

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Introduction

In representation theory, there are several localizations:

Example

- Serre quotient (abelian category)
- Verdier quotient (triangulated category)

etc.

In this talk, we introduce localization of extriangulated category, which covers the above examples.

Convention

We assume the following:

- All subcategories are full and closed under isomorphisms.
- All additive subcategories are closed under taking direct summands.

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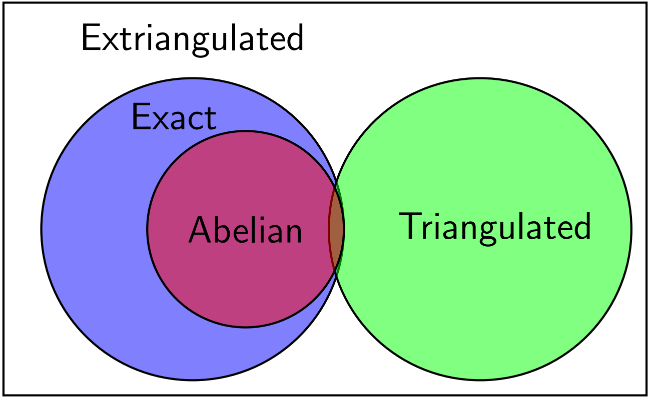
Extriangulated category

In this section, we introduce extriangulated categories which unifies exact categories and triangulated categories.



Exact category

Definition

An exact category consists of

- $\mathcal{C}$: an additive category
- $\mathcal{E}$: conflations, a class of kernel-cokernel pairs

Example

- extension-closed subcategories of abelian categories
- the category $C(\mathcal{A})$ of complexes of an abelian category $\mathcal{A}$ with the class of termwise split exact sequences

Triangulated category

Definition

A triangulated category consists of

- $\mathcal{C}$: an additive category
- Σ : an autoequivalence functor on $\mathcal{C}$
- $\mathcal{E}$: a class of distinguished triangles $X \to Y \to Z \to \Sigma X$

Example

- the stable category of a Frobenius exact category
- the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$

Exact categories and triangulated categories are equipped with a biadditive functor and a class of sequences of morphisms.

- In exact categories, a class of conflations defines a biadditive functror $\mathrm{Ext}^1_{\mathcal{C}}(-,-)$, the Yoneda extension group.
- In triangulated categories, any distinguished triangle is obtained from an element of a biadditive functor $\mathcal{C}(-,\Sigma-)$, that is, for any morphism $f \in \mathcal{C}(Z,\Sigma X)$, there is a distinguished triangle

$$X \to Y \to Z \xrightarrow{f} \Sigma X$$

by the axiom of triangulated categories.

Extriangulated category

Definition (Nakaoka-Palu '19)

An extriangulated category consists of

- $\mathcal{C}$: an additive category
- $\mathbb{E}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to Ab$: a biadditive functor
- $\mathfrak{s}$ assigns an equivalence class $\mathfrak{s}(\delta) = [X \to Y \to Z]$ to any $\delta \in \mathbb{E}(Z, X)$.

We call $\mathfrak{s}(\delta) = [X \xrightarrow{f} Y \xrightarrow{g} Z]$ an $\mathfrak{s}$-conflation and f, g an $\mathfrak{s}$-inflation, an $\mathfrak{s}$-deflation, respectively.

Extriangulated category

Example

- exact categories $\fallingdotseq$ extriangulated categories in which any $\mathfrak{s}$-inflation is mono and any $\mathfrak{s}$-deflation is epi
- triangulated categories $\fallingdotseq$ extriangulated categories in which any morphism is both an $\mathfrak{s}$-inflation and an $\mathfrak{s}$-deflation
- extension-closed subcategories of (ex)triangulated categories

Extriangulated(exact) functor

Definition (Bennett-Tennenhaus-Shah '21)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ and $(\mathcal{D}, \mathbb{F}, \mathfrak{t})$ be extriangulated categories. An additive functor $F\colon \mathcal{C} \to \mathcal{D}$ (with a natural transformation $\eta\colon \mathbb{E}(-,-) \to \mathbb{F}(F-, F-)$) is an extriangulated functor if it sends $\mathfrak{s}$-conflations to $\mathfrak{t}$-conflations.

Example

- exact functors $\fallingdotseq$ extriangulated functors between exact categories
- triangle functors $\fallingdotseq$ extriangulated functors between triangulated categories

Extriangulated(exact) functor

Example

- Let $\mathcal{T}$ be a triangulated category and $\mathcal{H}$ a heart of a t-structure. The inclusion functor $\mathcal{H} \to \mathcal{T}$ is an extriangulated functor.
- Let $\mathcal{E}$ be a Frobenius exact category and $\underline{\mathcal{E}}$ the stabe category of $\mathcal{E}$. The quotient functor $\mathcal{E} \to \underline{\mathcal{E}}$ is an extriangulated functor.

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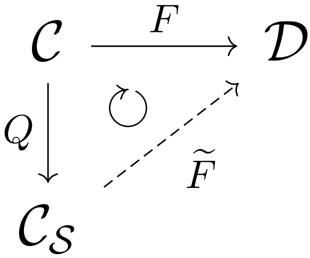
Localization

Definition

Let $\mathcal{C}$ be a category and $\mathcal{S}$ a class of morphisms in $\mathcal{C}$. The category $\mathcal{C}_{\mathcal{S}}$ and the functor $Q\colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ is a localization of $\mathcal{C}$ by $\mathcal{S}$ if it satisfies the following properties.

- For any $s \in \mathcal{S}$, $Q(s)$ is an isomorphism.
- For any category $\mathcal{D}$ and any functor $F\colon \mathcal{C} \to \mathcal{D}$ which sends all morphisms in $\mathcal{S}$ to isomorphisms, there exists a unique functor $\widetilde{F}\colon \mathcal{C}_{\mathcal{S}} \to \mathcal{D}$ such that $F = \widetilde{F} \circ Q$.



Localization of extriangulated category

Theorem (Nakaoka-Ogawa-S arXiv '21)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{S}$ a class of morphisms in $\mathcal{C}$. Suppose that $\mathcal{S}$ satisfies

- *$\mathcal{S}$ is a multiplicative system in $\mathcal{C}$.*
- *the two out of three property with respect to compositions.*
- *some compatibility conditions with repect to the extriangulated structure.*

Then there exists a localization $Q\colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$, and $\mathcal{C}_{\mathcal{S}}$ has the extriangulated structure such that Q is an extriangulated functor.

In the rest, we introduce localization of extriangulated categories by biresolving subcategories and percolating subcategories.

Biresolving subcategory

- biresolving subcategories in exact categories [Rump]
- Hovey twin cotorsion pairs [Nakaoka-Palu]
- Verdier quotient [Verdier]

Percolating subcatgory

- two-sided admissibly percolating subcategories in exact categories [Henrard-Roosmalen]
- Serre quotient [Gabriel]
- Verdier quotient [Verdier]

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Verdier quotient

Theorem (Verdier '67)

Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ a thick subcategory of $\mathcal{T}$. Then

1. $\mathcal{S}_{\mathcal{N}} := \{s \mid \mathrm{Cone}(s) \in \mathcal{N}\}$ *is a multiplicative system in $\mathcal{T}$, hence the localization $\mathcal{T}_{\mathcal{S}_{\mathcal{N}}}$ is an additive category and the localization functor $Q \colon \mathcal{T} \to \mathcal{T}_{\mathcal{S}_{\mathcal{N}}}$ is an additive functor.*
2. $\mathcal{T}_{\mathcal{S}_{\mathcal{N}}}$ *becomes a triangulated category.*
3. $Q \colon \mathcal{T} \to \mathcal{T}_{\mathcal{S}_{\mathcal{N}}}$ *is a triangulated functor.*

Biresolving subcategory

Let $\mathcal{E}$ be an exact category.

Definition (Rump '21)

A subcategory $\mathcal{A}$ of $\mathcal{E}$ is a biresolving subcategory if it satisfies

- the two out of three property with respect to any conflation in $\mathcal{E}$.
- For any $X \in \mathcal{E}$, there are an inflation $X \to A_1$ and an deflation $A_2 \to X$ with $A_1, A_2 \in \mathcal{A}$.

Example

Let $\mathcal{E}$ be a Frobenius exact category and $\mathcal{A}$ a subcategory consisting of projective and injective objects. Then $\mathcal{A}$ is a biresolving subcategory.

Biresolving subcategory

Theorem (Rump '21)

Let $\mathcal{E}$ be an exact category and $\mathcal{A}$ a biresolving subcategory of $\mathcal{E}$. Then

1. $\mathcal{S}_{\mathcal{A}} := \{s \mid \overline{s}$ *is monic and epic in the ideal quotient* $\mathcal{E}/\mathcal{A}\}$ *is a multiplicative system.*
2. $\mathcal{E}_{\mathcal{S}_{\mathcal{A}}}$ *is a triangulated category.*

This generalize a construction of triangulated structures on stable categories of Frobenius exact categories.

Biresolving subcategory

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

Definition

Let $\mathcal{N}$ be an additive subcategory of $\mathcal{C}$.

- $\mathcal{N}$ is a thick subcategory if it satisfies the two out of three property with respect to any $\mathfrak{s}$-conflation.
- Moreover $\mathcal{N}$ is called a biresolving subcategory if it satisfies the following condition.
  For any $X \in \mathcal{C}$, there are an $\mathfrak{s}$-inflation $X \to N_1$ and an $\mathfrak{s}$-deflation $N_2 \to X$ with $N_1, N_2 \in \mathcal{N}$.

Biresolving subcategory

Example

- The previous definition of biresolving subcategories is the same as in exact categories.
- In triangulated categories, the previous definition of thick subcategories coincides with usual one, and all thick subcategories are biresolving.

Main result

Theorem (Nakaoka-Ogawa-S arXiv '21)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{N}$ a biresolving subcategory of $\mathcal{C}$. Then

1. $\mathcal{S}_{\mathcal{N}} := \{\overset{f}{\rightarrowtail}\overset{g}{\twoheadrightarrow} \mid \mathrm{Cone}\, f, \mathrm{CoCone}\, g \in \mathcal{N}\}$ *gives a multiplicative system in the ideal quotient* $\mathcal{C}/\mathcal{N}$.
2. *the localization* $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ *is a triangulated category.*
3. *the localization functor* $Q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ *is an extriangulated functor.*

This result recovers the Verdier quotient and the Rump's result.

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Serre quotient

Theorem (Gabriel '62)

Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then

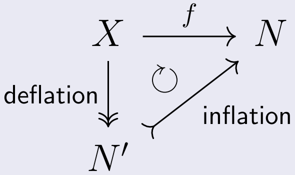
1. $\mathcal{S}_{\mathcal{B}} := \{s \mid \mathrm{Ker}(s), \mathrm{Coker}(s) \in \mathcal{B}\}$ *is a multiplicative system in* $\mathcal{A}$.
2. *the localization* $\mathcal{A}_{\mathcal{S}_{\mathcal{B}}}$ *becomes an abelian category.*
3. *the localization functor* $Q \colon \mathcal{A} \to \mathcal{A}_{\mathcal{S}_{\mathcal{B}}}$ *is an exact functor.*

Two-sided admissibly percolating subcategory

Definition (Henrard-Roosmalen arXiv '19)

Let $\mathcal{E}$ be an exact category. An additive subcategory $\mathcal{N}$ of $\mathcal{E}$ is a two-sided admissibly percolating subcategory if it satisfies

- $X, Z \in \mathcal{N}$ if and only if $Y \in \mathcal{N}$ for any conflation $X \rightarrowtail Y \twoheadrightarrow Z$.
- For any morphism $f\colon X \to N$ with $N \in \mathcal{N}$, there is a commutative diagram



with $N' \in \mathcal{N}$, and dual of this condition.

Two-sided admissibly percolating subcategory

Theorem (Henrard-Roosmalen arXiv '19)

Let $\mathcal{E}$ be an exact category and $\mathcal{N}$ a two-sided admissibly percolating subcategory of $\mathcal{E}$. Then

1. $\mathcal{S}_{\mathcal{N}} := \{\overset{f}{\twoheadrightarrow}\overset{g}{\rightarrowtail} \mid \operatorname{Ker} f, \operatorname{Coker} g \in \mathcal{N}\}$ *is a multiplicative system in* $\mathcal{E}$.
2. *the localization* $\mathcal{E}_{\mathcal{S}_{\mathcal{N}}}$ *becomes an exact category.*
3. *the localization functor* $Q\colon \mathcal{E} \to \mathcal{E}_{\mathcal{S}_{\mathcal{N}}}$ *is an exact functor.*

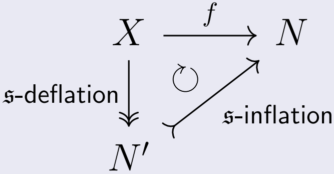
This covers Serre quotient in the special case.

Percolating subcategory

Definition

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. An additive subcategory $\mathcal{N}$ of $\mathcal{C}$ is a percolating subcategory if it satisfies

- the two out of three property with respect to any $\mathfrak{s}$-conflation in $\mathcal{C}$.
- For any morphism $f\colon X \to N$ with $N \in \mathcal{N}$, there is a commutative diagram



with $N' \in \mathcal{N}$, and dual of this condition.

Percolating subcategory

Example

- The previous definition of percolating subcategories is the same as in exact categories.
- In triangulated categories, percolating subcategories coincide with thick subcategories.

Main result

Theorem (Nakaoka-Ogawa-S arXiv '21)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{N}$ a percolating subcategory of $\mathcal{C}$ with some technical assumptions. Then

1. $\mathcal{S}_{\mathcal{N}} := \{ \overset{f}{\twoheadrightarrow} \overset{g}{\rightarrowtail} \mid \operatorname{CoCone} f, \operatorname{Cone} g \in \mathcal{N} \}$ *is a multiplicative system in* $\mathcal{C}$.
2. *the localization* $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ *becomes an extriangulated category.*
3. *the localization functor* $Q \colon \mathcal{C} \to \mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ *is an extriangulated functor.*
4. *If* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *is exact (abelian), then so is* $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$.

Summary

- The class of extriangulated categories includes exact categories, triangulated categories and their extension-closed subcategories.
- a thick subcategory $\mathcal{N}$ with some conditions
  $\Rightarrow$ the localization $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ is extriangulated
- The foundation of an extriangulated category has been completed.

Thank you for listening.

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