INTERVALS OF s-TORSION PAIRS IN EXTRIANGULATED CATEGORIES WITH NEGATIVE FIRST EXTENSIONS

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Abstract. In a triangulated category, for a given t-structure, the HRS-tilting induces an isomorphism between the poset of certain t-structures and the poset of torsion pairs in the heart of the t-structure. On the other hand, Asai–Pfeifer and Tattar established a poset isomorphism for torsion pairs in an abelian category. In this article, as a common generalization of t-structures and torsion pairs, we introduce the notion of s-torsion pairs in an extriangulated category with a negative first extension. Moreover, we provide a poset isomorphism for s-torsion pairs which unifies two poset isomorphisms above.

Throughout this article, we assume that every category is skeletally small, that is, the isomorphism classes of objects form a set. In addition, all subcategories are assumed to be full and closed under isomorphisms.

First we give the definition of torsion pairs in an exact category.

Definition 1. Let $\mathcal{E}$ be an exact category. A pair $(\mathcal{T}, \mathcal{F})$ of subcategories of $\mathcal{E}$ is called a *torsion pair* in $\mathcal{E}$ if it satisfies the following two conditions.

- For each $E \in \mathcal{E}$, there exists a conflation $0 \to T \to E \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- $\mathcal{E}(\mathcal{T}, \mathcal{F}) = 0$.

Let $\mathsf{tors}\,\mathcal{E}$ denote the set of torsion pairs in $\mathcal{E}$. We write $(\mathcal{T}_1, \mathcal{F}_1) \leq (\mathcal{T}_2, \mathcal{F}_2)$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Then $(\mathsf{tors}\,\mathcal{E}, \leq)$ clearly becomes a poset. Let $t_1 := (\mathcal{T}_1, \mathcal{F}_1)$ and $t_2 := (\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in $\mathcal{E}$ with $t_1 \leq t_2$. Let $\mathsf{tors}[t_1, t_2]$ denote the *interval* in the poset of torsion pairs in $\mathcal{E}$ consisting of $t := (\mathcal{T}, \mathcal{F})$ with $t_1 \leq t \leq t_2$. We call the subcategory $\mathcal{H}_{[t_1,t_2]} := \mathcal{T}_2 \cap \mathcal{F}_1$ the *heart* of $\mathsf{tors}[t_1, t_2]$. Since the heart $\mathcal{H}_{[t_1,t_2]}$ is an extension-closed subcategory, it becomes an exact category.

The following isomorohism induces fruitful results for the poset structure of torsion pairs in an abelian category.

Theorem 2 ([2, 7])**.** *Let $\mathcal{A}$ be an abelian category. For $i = 1, 2$, let $t_i := (\mathcal{T}_i, \mathcal{F}_i) \in \mathsf{tors}\,\mathcal{A}$ with $t_1 \leq t_2$. Then there exists a poset isomorphism between $\mathsf{tors}[t_1, t_2]$ and $\mathsf{tors}\,\mathcal{H}_{[t_1,t_2]}$.*

This isomorphism originally appeared in the context of τ-tilting reduction in [5].

Next we recall the definition of t-structures on a triangulated category.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 3. Let $\mathcal{D}$ be a triangulated category with a shift functor Σ. A pair $(\mathcal{U}, \mathcal{V})$ of subcategories of $\mathcal{D}$ is called a *t-structure* on $\mathcal{D}$ if it satisfies the following three conditions.

- For each $D \in \mathcal{D}$, there exists a triangle $U \to D \to V \to \Sigma U$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- $\mathcal{D}(\mathcal{U}, \mathcal{V}) = 0$.
- $\mathcal{U}$ is closed under a positive shift, that is, $\Sigma\mathcal{U} \subseteq \mathcal{U}$.

It is well known that the heart $\mathcal{U} \cap \Sigma\mathcal{V}$ of a t-structure $(\mathcal{U}, \mathcal{V})$ is always an abelian category. Let $\mathsf{t\text{-}str}\,\mathcal{D}$ denote the poset of t-structures on $\mathcal{D}$, where we define $(\mathcal{U}_1, \mathcal{V}_1) \leq (\mathcal{U}_2, \mathcal{V}_2)$ if $\mathcal{U}_1 \subseteq \mathcal{U}_2$. For t-structures $(\mathcal{U}_1, \mathcal{V}_1) \leq (\mathcal{U}_2, \mathcal{V}_2)$ on $\mathcal{D}$, let

$$\mathsf{t\text{-}str}[(\mathcal{U}_1, \mathcal{V}_1), (\mathcal{U}_2, \mathcal{V}_2)] := \{(\mathcal{U}, \mathcal{V}) \in \mathsf{t\text{-}str}\,\mathcal{D} \mid \mathcal{U}_1 \subseteq \mathcal{U} \subseteq \mathcal{U}_2\}.$$

Happel, Reiten and Smalø ([4]) provided a construction of new t-structures through torsion pairs in the heart of a given t-structure. This construction induces a close connection between t-structures and torsion pairs as follows.

Theorem 4 ([4, 8])**.** *Let $\mathcal{D}$ be a triangulated category with a shift functor Σ. Let $(\mathcal{U}, \mathcal{V}) \in \mathsf{t\text{-}str}\,\mathcal{D}$ and $\mathcal{H} := \mathcal{U} \cap \Sigma\mathcal{V}$ the heart of $(\mathcal{U}, \mathcal{V})$. Then there exists a poset isomorphism between $\mathsf{t\text{-}str}[(\Sigma\mathcal{U}, \Sigma\mathcal{V}), (\mathcal{U}, \mathcal{V})]$ and $\mathsf{tors}\,\mathcal{H}$.*

The aim of this article is to show that two poset isomorphisms in Theorem 2 and Theorem 4 are consequences of a more general poset isomorphism in an extriangulated category, which is a simultaneous generalization of triangulated categories and exact categories.

Let $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ denote an extriangulated category. For definition and terminologies, see [6]. A complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ is called an $\mathfrak{s}$-*conflation* if there exists $\delta \in \mathbb{E}(C, A)$ such that $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$, where $[A \xrightarrow{f} B \xrightarrow{g} C]$ is an equivalence class of a complex $A \xrightarrow{f} B \xrightarrow{g} C$. We write the $\mathfrak{s}$-conflation as $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$. For two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{C}$, let $\mathcal{X} * \mathcal{Y}$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $X \to M \to Y \dashrightarrow$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is said to be *extension-closed* if $\mathcal{C}' * \mathcal{C}' \subseteq \mathcal{C}'$.

We introduce a negative first extension structure on an extriangulated category.

Definition 5 ([1, Definition 2.3])**.** Let $\mathcal{C}$ be an extriangulated category. A *negative first extension structure* on $\mathcal{C}$ consists of the following data:

(NE1) $\mathbb{E}^{-1} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}b$ is an additive bifunctor.

(NE2) For each $\delta \in \mathbb{E}(C, A)$, there exist two natural transformations

$$\delta_\sharp^{-1} : \mathbb{E}^{-1}(-, C) \to \mathcal{C}(-, A),$$
$$\delta_{-1}^\sharp : \mathbb{E}^{-1}(A, -) \to \mathcal{C}(C, -)$$

such that for each $\mathfrak{s}$-conflation $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ and each $W \in \mathcal{C}$, two sequences

$$\mathbb{E}^{-1}(W, A) \xrightarrow{\mathbb{E}^{-1}(W,f)} \mathbb{E}^{-1}(W, B) \xrightarrow{\mathbb{E}^{-1}(W,g)} \mathbb{E}^{-1}(W, C) \xrightarrow{(\delta_\sharp^{-1})_W} \mathcal{C}(W, A) \xrightarrow{\mathcal{C}(W,f)} \mathcal{C}(W, B),$$

$$\mathbb{E}^{-1}(C, W) \xrightarrow{\mathbb{E}^{-1}(g,W)} \mathbb{E}^{-1}(B, W) \xrightarrow{\mathbb{E}^{-1}(f,W)} \mathbb{E}^{-1}(A, W) \xrightarrow{(\delta_{-1}^\sharp)_W} \mathcal{C}(C, W) \xrightarrow{\mathcal{C}(g,W)} \mathcal{C}(B, W)$$

are exact.

Then we call $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s}, \mathbb{E}^{-1})$ an *extriangulated category with a negative first extension.*

Note that a negative first extension is a special case of partial δ-functors in the sense of [3, Definition 4.7]. Triangulated categories and exact categories naturally admit negative first extension structures as follows.

Example 6. (1) A triangulated category $\mathcal{D}$ becomes an extriangulated category with a negative first extension by the following data.

- $\mathbb{E}(C, A) := \mathcal{D}(C, \Sigma A)$ for all $A, C \in \mathcal{D}$, where Σ is a shift functor of $\mathcal{D}$.
- For $\delta \in \mathbb{E}(C, A)$, we take a triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} \Sigma A$. Then we define $\mathfrak{s}(\delta) := [A \xrightarrow{f} B \xrightarrow{g} C]$.
- $\mathbb{E}^{-1}(C, A) := \mathcal{D}(C, \Sigma^{-1} A)$ for all $A, C \in \mathcal{D}$.
- For an $\mathfrak{s}$-conflation $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$, we define two natural transformations $\delta_\sharp^{-1}$ and $\delta^\sharp_{-1}$ as follows: for $W \in \mathcal{D}$,

$$(\delta_\sharp^{-1})_W : \mathbb{E}^{-1}(W, C) = \mathcal{D}(W, \Sigma^{-1} C) \xrightarrow{\mathcal{D}(W, \Sigma^{-1}\delta)} \mathcal{D}(W, A),$$
$$(\delta^\sharp_{-1})_W : \mathbb{E}^{-1}(A, W) = \mathcal{D}(A, \Sigma^{-1} W) \cong \mathcal{D}(\Sigma A, W) \xrightarrow{\mathcal{D}(\delta, W)} \mathcal{D}(C, W).$$

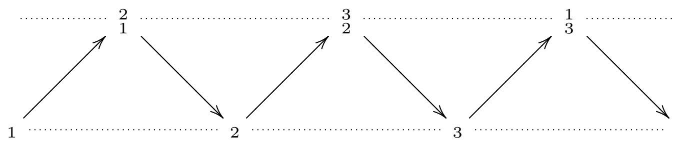
(2) An exact category $\mathcal{E}$ becomes an extriangulated category with a negative first extension by the following data.

- $\mathbb{E}(C, A)$ is the set of isomorphism classes of conflations in $\mathcal{E}$ of the form $0 \to A \to B \to C \to 0$ for $A, C \in \mathcal{E}$.
- $\mathfrak{s}$ is the identity.
- $\mathbb{E}^{-1}(C, A) = 0$ for all $A, C \in \mathcal{E}$.
- For each $W \in \mathcal{E}$, the maps $(\delta_\sharp^{-1})_W$ and $(\delta^\sharp_{-1})_W$ are zero.

(3) Let $\mathcal{C}$ be an extriangulated category with a negative first extension and let $\mathcal{C}'$ be an extension-closed subcategory of $\mathcal{C}$. Then by restricting the extriangulated structure and the negative first extension structure to $\mathcal{C}'$, we can regard $\mathcal{C}'$ as an extriangulated category with a negative first extension.

The following example shows that negative first extension structures are not uniquely determined by given extriangulated categories.

Example 7. Let $\mathbf{k}$ be an algebraically closed field. Consider the stable category $\mathcal{D} := \underline{\mathsf{mod}}\,\Lambda$ of a self-injective Nakayama $\mathbf{k}$-algebra Λ with three simple modules and the Loewy length three. Then the Auslander-Reiten quiver of $\mathcal{D}$ is as follows, where two 1's are identified.



Since the subcategory $\mathcal{A} := \mathsf{add}\{1, {}^2_1, 2\}$ is clearly equivalent to the category of finite-dimensional representations of an A_2 quiver, it is abelian. Thus $\mathcal{A}$ becomes an extriangulated category with a negative first extension $\mathbb{E}_1^{-1} := 0$ by Example 6(2). On the other hand, since $\mathcal{A}$ is extension-closed in $\mathcal{D}$, it becomes an induced extriangulated category with a negative first extension $\mathbb{E}_2^{-1}(-,-) := \mathcal{D}(-,\Sigma^{-1}-)$ by Example 6(3). We can check that extriangulated category structures coincide with each other, but negative first extension structures do not. Indeed, $\mathbb{E}_2^{-1}(2,1) = \mathcal{D}(2,\Sigma^{-1}(1)) = \mathcal{D}(2,\substack{3\\2}) \neq 0$ holds.

In the following, let $\mathcal{C} := (\mathcal{C},\mathbb{E},\mathfrak{s},\mathbb{E}^{-1})$ be an extriangulated category with a negative first extension. We introduce the notion of s-torsion pairs in $\mathcal{C}$.

Definition 8. Let $\mathcal{C}$ be an extriangulated category with a negative first extension. We call a pair $(\mathcal{T},\mathcal{F})$ of subcategories of $\mathcal{C}$ an *s-torsion pair* in $\mathcal{C}$ if it satisfies the following three conditions.

(STP1) $\mathcal{C} = \mathcal{T} * \mathcal{F}$.
(STP2) $\mathcal{C}(\mathcal{T},\mathcal{F}) = 0$.
(STP3) $\mathbb{E}^{-1}(\mathcal{T},\mathcal{F}) = 0$.

Let $\mathsf{tors}\,\mathcal{C}$ denote the poset of s-torsion pairs in $\mathcal{C}$, where we define $(\mathcal{T}_1,\mathcal{F}_1) \leq (\mathcal{T}_2,\mathcal{F}_2)$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

The following examples show that s-torsion pairs are a common generalization of t-structures on a triangulated category and torsion pairs in an exact category.

Example 9. (1) Let $\mathcal{D}$ be a triangulated category. By regarding $\mathcal{D}$ as the extriangulated category with the negative first extension (see Example 6(1)), t-structures on $\mathcal{D}$ are exactly s-torsion pairs in $\mathcal{D}$, that is, $\mathsf{t\text{-}str}\,\mathcal{D} = \mathsf{stors}\,\mathcal{D}$. Indeed, let $(\mathcal{U},\mathcal{V})$ be a pair of subcategories of $\mathcal{D}$ satisfying the conditions (STP1) and (STP2). By the negative first extension structure on $\mathcal{D}$, we have $\mathbb{E}^{-1}(\mathcal{U},\mathcal{V}) = \mathcal{D}(\mathcal{U},\Sigma^{-1}\mathcal{V}) \cong \mathcal{D}(\Sigma\mathcal{U},\mathcal{V})$. Hence $\mathbb{E}^{-1}(\mathcal{U},\mathcal{V}) = 0$ if and only if $\Sigma\mathcal{U} \subseteq \{X \in \mathcal{D} \mid \mathcal{D}(X,\mathcal{V}) = 0\} = \mathcal{U}$.

(2) Let $\mathcal{E}$ be an exact category. By regarding $\mathcal{E}$ as the extriangulated category with the negative first extension (see Example 6(2)), it follows from $\mathbb{E}^{-1} = 0$ that torsion pairs in the exact category $\mathcal{E}$ are exactly s-torsion pairs in $\mathcal{E}$, that is, we have $\mathsf{tors}\,\mathcal{E} = \mathsf{stors}\,\mathcal{E}$.

Taking negative first extension structures different from Example 6(2), we give an example which satisfies (STP1) and (STP2) but does not satisfy (STP3).

Example 10. Let Λ and $\mathcal{A}$ be in Example 7. Due to Example 7, we regard $\mathcal{A}$ as the extriangulated category with the negative first extension $\mathbb{E}_2^{-1}$. Since $\mathcal{A}$ is an abelian category, a pair of subcategories $(\mathcal{T},\mathcal{F})$ satisfies (STP1) and (STP2) if and only if it is a (usual) torsion pair in the abelian category $\mathcal{A}$. Thus, $(\mathsf{add}\{\substack{2\\1},2\},\mathsf{add}\{1\})$ satisfies (STP1) and (STP2). On the other hand, since $\mathbb{E}_2^{-1}(2,1) \neq 0$ holds, this pair does not satisfy (STP3).

The following notion plays an important role in this article.

Definition 11. Let $\mathcal{C}$ be an extriangulated category with a negative first extension. For $i = 1,2$, let $t_i := (\mathcal{T}_i,\mathcal{F}_i) \in \mathsf{stors}\,\mathcal{C}$ with $t_1 \leq t_2$. Then we call the subposet

$$\mathsf{stors}[t_1,t_2] := \{t := (\mathcal{T},\mathcal{F}) \in \mathsf{stors}\,\mathcal{C} \mid t_1 \leq t \leq t_2\} \subseteq \mathsf{stors}\,\mathcal{C}$$

an *interval* in $\mathsf{stors}\,\mathcal{C}$ and the subcategory $\mathcal{H}_{[t_1,t_2]} := \mathcal{T}_2 \cap \mathcal{F}_1 \subseteq \mathcal{C}$ the *heart* of the interval $\mathsf{stors}[t_1,t_2]$. Since $\mathcal{H}_{[t_1,t_2]}$ is extension-closed, we can regard $\mathcal{H}_{[t_1,t_2]}$ as the extriangulated category with the negative first extension (see Example 6(3)).

By Example 9(1), we can easily check that the heart of a t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathcal{D}$ coincides with the heart of the interval $\mathsf{stors}[(\Sigma\mathcal{U}, \Sigma\mathcal{V}), (\mathcal{U}, \mathcal{V})]$.

Now we state a main result of this article.

Theorem 12 ([1, Theorem 3.9])**.** *Let $\mathcal{C}$ be an extriangulated category with a negative first extension. For $i = 1, 2$, let $t_i := (\mathcal{T}_i, \mathcal{F}_i) \in \mathsf{stors}\,\mathcal{C}$ with $t_1 \leq t_2$. Then there exist mutually inverse poset isomorphisms*

$$\mathsf{stors}\,[t_1, t_2] \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{stors}\,\mathcal{H}_{[t_1,t_2]},$$

where $\Phi(\mathcal{T}, \mathcal{F}) := (\mathcal{T} \cap \mathcal{F}_1, \mathcal{T}_2 \cap \mathcal{F})$ and $\Psi(\mathcal{X}, \mathcal{Y}) := (\mathcal{T}_1 * \mathcal{X}, \mathcal{Y} * \mathcal{F}_2)$. In particular, Φ and Ψ preserve hearts, that is, for $\mathsf{stors}[t, t'] \subseteq \mathsf{stors}[t_1, t_2]$ and $\mathsf{stors}[x, x'] \subseteq \mathsf{stors}\,\mathcal{H}_{[t_1,t_2]}$, we have $\mathcal{H}_{[t,t']} = \mathcal{H}_{[\Phi(t),\Phi(t')]}$ and $\mathcal{H}_{[x,x']} = \mathcal{H}_{[\Psi(x),\Psi(x')]}$.

We give two applications of Theorem 12. We have the following result, which recovers Theorem 4.

Corollary 13. *Let $\mathcal{D}$ be a triangulated category. For $i = 1, 2$, let $(\mathcal{U}_i, \mathcal{V}_i) \in \mathsf{t\text{-}str}\,\mathcal{D}$ with $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and $\mathcal{H} := \mathcal{U}_2 \cap \mathcal{V}_1$. Then there exist mutually inverse poset isomorphisms*

$$\mathsf{t\text{-}str}\,[(\mathcal{U}_1, \mathcal{V}_1), (\mathcal{U}_2, \mathcal{V}_2)] \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{stors}\,\mathcal{H},$$

where $\Phi(\mathcal{T}, \mathcal{F}) := (\mathcal{T} \cap \mathcal{V}_1, \mathcal{U}_2 \cap \mathcal{F})$ and $\Psi(\mathcal{X}, \mathcal{Y}) := (\mathcal{U}_1 * \mathcal{X}, \mathcal{Y} * \mathcal{V}_2)$. In addition, if $\Sigma\mathcal{U}_2 \subseteq \mathcal{U}_1$ holds, then $\mathcal{H}$ becomes an exact category by the induced extriangulated structure, and we have $\mathsf{stors}\,\mathcal{H} = \mathsf{tors}\,\mathcal{H}$.

By Theorem 12, we have the following corollary, which is a further generalization of Theorem 2, where the abelian category case is proved.

Corollary 14. *Let $\mathcal{E}$ be an exact category. For $i = 1, 2$, let $t_i := (\mathcal{T}_i, \mathcal{F}_i) \in \mathsf{tors}\,\mathcal{E}$ with $t_1 \leq t_2$. Then there exist mutually inverse poset isomorphisms*

$$\mathsf{tors}\,[t_1, t_2] \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{tors}\,\mathcal{H}_{[t_1,t_2]},$$

where $\Phi(\mathcal{T}, \mathcal{F}) := (\mathcal{T} \cap \mathcal{F}_1, \mathcal{T}_2 \cap \mathcal{F})$ and $\Psi(\mathcal{X}, \mathcal{Y}) := (\mathcal{T}_1 * \mathcal{X}, \mathcal{Y} * \mathcal{F}_2)$.

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