EXAMPLES OF TILTING-DISCRETE SELF-INJECTIVE ALGEBRAS

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Abstract. In this note, we give two examples of tilting-discrete self-injective algebras which are not silting-discrete.

Notation. Throughout this note, $\mathbb{k}$ is an algebraically closed field and $\mathbb{D} := \mathrm{Hom}_{\mathbb{k}}(-, \mathbb{k})$. Let A be a basic connected finite dimensional $\mathbb{k}$-algebra and let $\mathcal{K}_A$ denote the bounded homotopy category of finitely generated projective A-modules with shift functor Σ.

Silting mutations. Recall the definition of silting mutations. We refer to [4].

An object $M \in \mathcal{K}_A$ is called a *silting object* if $\mathrm{Hom}_{\mathcal{K}_A}(M, \Sigma^i M) = 0$ for all $i > 0$ and $\mathcal{K}_A = \mathsf{thick} M$, where $\mathsf{thick} M$ is the smallest triangulated subcategory of $\mathcal{K}_A$ containing M and closed under direct summands. We denote by $\mathrm{silt}\, A$ the set of isomorphism classes of basic silting objects in $\mathcal{K}_A$. By the definition, A is a silting object in $\mathcal{K}_A$.

Let $M = X \oplus N$ be a basic silting object in $\mathcal{K}_A$. Take a minimal left $\mathsf{add} N$-approximation $f : X \to N'$ and a triangle $X \xrightarrow{f} N' \to Y \to \Sigma X$. Then $\mu_X(M) := Y \oplus N$ is a basic silting object in $\mathcal{K}_A$ and called a (*left*) *silting mutation* of M with respect to X. If X is indecomposable, we call $\mu_X(M)$ an *irreducible silting mutation*.

Silting-discrete algebras. Recall the definition of silting-discrete algebras. We refer to [3, 5]. For two objects $M, N \in \mathcal{K}_A$, we write $M \geq N$ if $\mathrm{Hom}_{\mathcal{K}_A}(M, \Sigma^i N) = 0$ for all $i > 0$.

An algebra A is called a *silting-discrete algebra* if for each integer $d > 0$, the set

$$(d+1)\text{-}\mathrm{silt}\, A := \{N \in \mathrm{silt}\, A \mid A \geq N \geq \Sigma^d A\}$$

is finite. Note that if A is a silting-discrete algebra, then all silting objects in $\mathcal{K}_A$ are obtained by a finite sequence of irreducible silting mutations from A (up to shift). If A is a local algebra, then

$$\mathrm{silt}\, A = \{\Sigma^i A \mid i \in \mathbb{Z}\}.$$

Hence local algebras are silting-discrete.

We give a characterization of a finite dimensional algebra to be silting-discrete.

Proposition 1 ([5, Theorem 2.4])**.** *An algebra A is silting-discrete if and only if for each silting object M given by iterated irreducible silting mutation from A, the set*

$$2_M\text{-}\mathrm{silt}\, A := \{N \in \mathrm{silt}\, A \mid M \geq N \geq \Sigma M\}$$

is finite.

The detailed version of this paper will be submitted for publication elsewhere.

Tilting mutations for self-injective algebras. Recall the definition of tilting mutations. We refer to [9].

An object $M \in \mathcal{K}_A$ is called a *tilting object* if $\mathrm{Hom}_{\mathcal{K}_A}(M, \Sigma^i M) = 0$ for all $i \neq 0$ and $\mathcal{K}_A = \mathsf{thick} M$. We denote by $\mathrm{tilt}\, A$ the set of isomorphism classes of basic tilting objects in $\mathcal{K}_A$. Clearly, A is a tilting object in $\mathcal{K}_A$.

By the definition, tilting objects are silting objects. Remark that silting mutations of tilting objects are silting objects but not necessarily tilting objects. However, if A is self-injective, then it is known that special silting mutations of tilting objects are also tilting objects.

In the following, we assume that A is a self-injective algebra. Then the Nakayama functor $\nu_A := - \otimes_A \mathbb{D}A$ gives an auto-equivalence of $\mathcal{K}_A$. By [6] (and also [3, Theorem A.4]), a silting object $M \in \mathcal{K}_A$ is tilting if and only if it is ν_A-stable (i.e., $\nu_A M \cong M$).

Let $M = X \oplus N$ be a basic tilting object in $\mathcal{K}_A$ with X ν_A-stable. Note that N is also a ν_A-stable object. Then $\mu_X(M)$ is a ν_A-stable silting object, and hence it is a tilting object. We call $\mu_X(M)$ a *tilting mutation* if X is a ν_A-stable object. Moreover, it is said to be *irreducible* if X is non-zero and satisfies the property that, if $X' \neq 0$ is a ν_A-stable direct summand of X, then $X' = X$.

Tilting-discrete self-injective algebras. Recall the definition of tilting-discrete algebras. We refer to [5].

An algebra A is called a *tilting-discrete algebra* if for each integer $d > 0$, the set

$$(d+1)\text{-}\mathrm{tilt}\, A := \mathrm{tilt}\, A \cap (d+1)\text{-}\mathrm{silt}\, A$$

is finite. Note that if A is a tilting-discrete self-injective algebra, then all tilting objects in $\mathcal{K}_A$ are obtained by a finite sequence of irreducible tilting mutations from A (up to shift).

We give an example of tilting-discrete self-injective algebras. The Nakayama functor ν_A is said to be *cyclic* if it acts transitively on the set of isomorphism classes of indecomposable projective A-modules.

Proposition 2 ([1, Proposition 2.14])**.** *Let A be a self-injective algebra. If ν_A is cyclic, then*

$$\mathrm{tilt}\, A = \{\Sigma^i A \mid i \in \mathbb{Z}\}.$$

In particular, A is tilting-discrete.

We give a characterization of a finite dimensional self-injective algebra to be tilting-discrete.

Proposition 3 ([5, Corollary 2.11])**.** *A self-injective algebra A is a tilting-discrete algebra if and only if for each tilting object M given by iterated irreducible tilting mutation from A, the set* $2\text{-}\mathrm{tilt}\, \mathrm{End}_{\mathcal{K}_A}(M)$ *is finite.*

The first example: trivial tilting-discrete case. Our aim of this note is to give an example of a self-injective algebra A satisfying the following properties.

- $\mathrm{tilt}\, A = \{\Sigma^i A \mid i \in \mathbb{Z}\}$.
- A is not silting-discrete.

Let $Q = (Q_0, Q_1)$ be a connected finite quiver, where Q_0 is the vertex set and Q_1 is the arrow set. Let $\Bbbk Q_l$ denote the subspace of the path algebra $\Bbbk Q$ generated by all paths of length of l. Define a new quiver $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1)$ as $\widetilde{Q}_0 := Q_0$ and $\widetilde{Q}_1 := Q_1^+ \coprod Q_1^-$, where $Q_1^+ := \{a^+ \mid a \in Q_1\}$ and $Q_1^- := \{a^- \mid a \in Q_1\}$. The correspondences $a \mapsto a^{\pm}$ induce $\Bbbk$-linear isomorphisms $(-)^{\pm} : \Bbbk Q_1 \to \Bbbk Q_1^{\pm}$ and moreover, they are extended to $\Bbbk$-linear isomorphisms $(-)^{\pm} : \oplus_{l\geq 1}\Bbbk Q_l \to \oplus_{l \geq 1} \Bbbk Q_l^{\pm}$.

Let $\Lambda := \Bbbk Q/I$ be a non-local self-injective algebra, where I is an admissible ideal of $\Bbbk Q$. Define a subspace $\widetilde{I}$ of $\Bbbk\widetilde{Q}$ as $\widetilde{I} := I^+ + I^- + I^d + I^c$, where $I^d = \langle a^+b^-, a^-b^+ \mid a, b \in Q_1\rangle$ and $I^c = \langle s^+ - s^- \mid s \in \operatorname{soc}\Lambda\rangle$. Since Λ is self-injective, $\widetilde{I}$ is a two-sided ideal of $\Bbbk\widetilde{Q}$. Moreover, if $\operatorname{soc}(e_i\Lambda) \subset \operatorname{rad}^2\Lambda$ holds for each $i \in Q_0$, then $\widetilde{I}$ is admissible.

We give an example of $\widetilde{Q}$ and $\widetilde{I}$.

Example 4. Let $Q := 1 \underset{b}{\overset{a}{\rightleftarrows}} 2$ and $I = \langle abab, baba\rangle$. Note that $\Lambda := \Bbbk Q/I$ is a self-injective algebra. Then we have

$$\widetilde{Q} = 1 \underset{b^-,\ b^+}{\overset{a^+,\ a^-}{\rightleftarrows}} 2$$

and

$$\begin{aligned}\widetilde{I} = \langle a^+b^+a^+b^+,\ b^+a^+b^+a^+,\ a^-b^-a^-b^-,\ b^-a^-b^-a^-,\ a^+b^-,\ a^-b^+,\\ b^+a^-,\ b^-a^+,\ a^+b^+a^+ - a^-b^-a^-,\ b^+a^+b^+ - b^-a^-b^-\rangle.\\ = \langle a^+b^-,\ a^-b^+,\ b^+a^-, b^-a^+,\ a^+b^+a^+ - a^-b^-a^-,\ b^+a^+b^+ - b^-a^-b^-\rangle.\end{aligned}$$

In the following, we assume that $\operatorname{soc}(e_i\Lambda) \subset \operatorname{rad}^2\Lambda$ holds for each $i \in Q_0$. The bound quiver algebra $A := \Bbbk\widetilde{Q}/\widetilde{I}$ has the following properties.

Proposition 5. *The algebra A is a basic self-injective algebra. Moreover, ν_Λ is cyclic if and only if ν_A is cyclic.*

The following theorem is one of main results of this note.

Theorem 6. *Assume that ν_Λ is cyclic. Then the following statements hold.*

(1) $\operatorname{tilt} A = \{\Sigma^i A \mid i \in \mathbb{Z}\}$. *In particular, A is a tilting-discrete algebra.*

(2) *A is not a silting-discrete algebra.*

Proof. By Proposition 5, A is a self-injective algebra.

(1) By Proposition 5, ν_A is cyclic. Thus the assertion follows from Proposition 2.

(2) Since $\widetilde{I}$ is admissible, A contains the path algebra of the Kronecker quiver as a factor algebra. Thus it follows from [10, Corollary 1.9] that $2\text{-}\operatorname{silt} A$ is not finite. Hence A is not a silting-discrete algebra. □

The second example: non-trivial tilting-discrete case. For integers $i \leq j$, let $[i, j] := \{i, i+1, \ldots, j-1, j\}$. Let n, m be positive integers. Define a quiver $\mathbb{T}_{n,m} := (\mathbb{T}_0, \mathbb{T}_1)$, where $\mathbb{T}_0$ is the vertex set and $\mathbb{T}_1$ is the arrow set, as follows:

- $\mathbb{T}_0 := \{(i,r) \mid i \in [1,n],\ r \in \mathbb{Z}/m\mathbb{Z}\}$,
- $\mathbb{T}_1 := \{a_{i,r} : (i,r) \to (i+1,r) \mid i \in [1,n-1],\ r \in \mathbb{Z}/m\mathbb{Z}\}$
$\coprod \{b_{i,r} : (i,r) \to (i-1,r+1) \mid i \in [2,n],\ r \in \mathbb{Z}/m\mathbb{Z}\}$.

Formally, put $a_{0,r} = a_{n,r} = b_{1,r} = b_{n+1,r} = 0$ for all $r \in \mathbb{Z}/m\mathbb{Z}$. We define an algebra $A_{n,m}$ as the bound quiver algebra $\Bbbk\mathbb{T}_{n,m}/I$, where I is the two-sided ideal generated by $a_{i,r}b_{i+1,r} - b_{i,r}a_{i-1,r+1}$ for all $i \in [1,n]$ and $r \in \mathbb{Z}/m\mathbb{Z}$. Note that $A_{n,m}$ is a self-injective algebra (see [7, 8]). Then we have the following theorem, which is a main result of this note.

Theorem 7. *Let $n, m \geq 5$ be integers satisfying $\gcd(n-1,m) = 1$. Assume that n is an odd number and m is not divisible by the characteristic of $\Bbbk$. Then $A_{n,m}$ is a tilting-discrete algebra but not silting-discrete.*

To prove Theorem 7, we need the following two propositions (for the proofs, see [2]).

Proposition 8. *Let n, m be positive integers. Then the following statements hold.*

1. *Assume that $n, m \geq 5$. Then $2\text{-silt}\, A_{n,m}$ is not a finite set. In particular, $A_{n,m}$ is not a silting-discrete algebra.*
2. *Assume that $\gcd(n-1,m) = 1$ and m is not divisible by the characteristic of $\Bbbk$. Then $2\text{-tilt}\, A_{n,m}$ is a finite set.*

Proposition 9. *Assume that $\gcd(n-1,m) = 1$ and n is an odd number. If M is a tilting object in $\mathcal{K}_{A_{n,m}}$ given by iterated irreducible tilting mutation from $A_{n,m}$, then the endomorphism algebra $\operatorname{End}_{\mathcal{K}_{A_{n,m}}}(M)$ is isomorphic to $A_{n,m}$.*

We give a proof of Theorem 7.

Proof of Theorem 7. By Proposition 8(1), $A_{n,m}$ is not a silting-discrete algebra. We show that $A_{n,m}$ is a tilting-discrete algebra. By Proposition 3, it is enough to show that for each tilting object M given by iterated irreducible tilting mutation from $A_{n,m}$, the set $2\text{-tilt}\operatorname{End}_{\mathcal{K}_{A_{n,m}}}(M)$ is a finite. By Proposition 9, the endomorphism algebra $\operatorname{End}_{\mathcal{K}_{A_{n,m}}}(M)$ is isomorphic to $A_{n,m}$. Hence it follows from Proposition 8(2) that $2\text{-tilt}\operatorname{End}_{\mathcal{K}_{A_{n,m}}}(M)$ is a finite set. Thus $A_{n,m}$ is tilting-discrete. □

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