τ-TILTING FINITE TRIANGULAR MATRIX ALGEBRAS

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Abstract. In this note, we discuss the τ-tilting finiteness of second triangular matrix algebras. Moreover, we give a new construction of silting-discrete algebras.

1. Introduction

Adachi, Iyama and Reiten [1] introduced the τ-tilting theory as a generalization of the classical tilting theory in terms of mutations. Support τ-tilting modules play a central role in τ-tilting theory, in fact, these modules are related to other contents of representation theory, such as torsion classes, silting objects and t-structures. In this note, we consider the finiteness of support τ-tilting modules that is called τ-tilting finite.

Throughout this note, algebras are always assumed to be finite dimensional over an algebraically closed field K. Modules are finite dimensional and right modules. For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ ($\mathsf{proj}\,\Lambda$) the category of (projective) modules over Λ. The perfect derived category of Λ is denoted by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$.

2. Preliminary

In this section, we recall the definition of support τ-tilting modules and silting objects. For a module M, $|M|$ denotes the number of pairwise non-isomorphic indecomposable direct summands of M.

2.1. **Support τ-tilting modules.** We denote by $\tau = \tau_\Lambda$ the Auslander–Reiten translation.

Definition 1. (1) A module M is said to be *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.
(2) A *τ-tilting module* M is defined to be τ-rigid with $|M| = |\Lambda|$.
(3) We say that a module M is *support τ-tilting* if there is an idempotent e of Λ such that M is a τ-tilting $\Lambda/(e)$-module.
(4) A module M is called *brick* if $\mathrm{End}_\Lambda(M) \simeq K$.
(5) An algebra Λ is said to be *τ-tilting finite* if Λ has only finitely many isomorphism classes of basic τ-tilting modules.

We denote by $\mathsf{s\tau\text{-}tilt}\,\Lambda$ (resp. $\mathsf{brick}\,\Lambda$) the set of isomorphism classes of basic support τ-tilting modules (resp. bricks).

Proposition 2. *For an algebra Λ, the following are equivalent.*
(1) Λ is τ-tilting finite;
(2) $\mathsf{s\tau\text{-}tilt}\,\Lambda$ is finite set;
(3) $\mathsf{brick}\,\Lambda$ is finite set.

The detailed version of this paper has been submitted for publication elsewhere.

2.2. **Silting objects.** Let us recall the definition of silting objects.

Definition 3. Let Λ be an algebra and T an object of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$.

(1) T is said to be *presilting* if it satisfies $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(T, T[i]) = 0$ for any positive integer $i > 0$.

(2) T is called *silting* if it is presilting and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda) = \mathsf{thick}\,T$. Here, $\mathsf{thick}\,T$ stands for the smallest thick subcategory of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$ containing T.

We denote by $\mathsf{silt}\,\Lambda$ the set of isomorphism classes of basic silting objects of $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$.

Definition 4. (1) For $T, U \in \mathsf{silt}\,\Lambda$, we write $T \geq U$ if $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)}(T, U[> 0]) = 0$.

(2) For $d > 0$,

$$\mathsf{d}\text{-}\mathsf{silt}\,\Lambda := \{T \in \mathsf{silt}\,\Lambda \mid \Lambda \geq T \geq \Lambda[d-1]\}.$$

(3) An algebra Λ is called *silting-discrete* if $\mathsf{d}\text{-}\mathsf{silt}\,\Lambda$ is finite set for any $d > 0$

Proposition 5. [1]

(1) $2\text{-}\mathsf{silt}\Lambda$ *is isomorphic to* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$.

(2) If Λ is silting-discrete, then Λ is τ-tilting finite.

3. Second triangular matrix algebras

The first aim of this section is to develop the Auslander–Reiten's results in [3] to the τ-tilting finiteness.

A main algebra we study here is the $n \times n$ upper triangular matrix algebra $T_n(\Lambda)$, which is isomorphic to $\Lambda \otimes_K K\overrightarrow{A_n}$. Here, $\overrightarrow{A_n}$ denotes the linearly oriented A_n-quiver $1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$. As is well-known, we can identify the category $\mathsf{mod}\,T_2(\Lambda)$ with the category of homomorphisms in $\mathsf{mod}\,\Lambda$; that is, the objects are triples (M, N, f) of Λ-modules M, N and a Λ-homomorphism $f : M \to N$. A morphism $(M_1, N_1, f_1) \to (M_2, N_2, f_2)$ is a pair (α, β) of Λ-homomorphisms $\alpha : M_1 \to M_2$ and $\beta : N_1 \to N_2$ satisfying $f_2 \circ \alpha = \beta \circ f_1$.

For an additive category $\mathcal{C}$, we denote by $\mathsf{mod}\,\mathcal{C}$ the full subcategory of the functor category of $\mathcal{C}$ consisting of finitely generated functors.

Inspired by [3, Theorem 1.1], we have the second main result of this paper.

Theorem 6. *Assume that Λ is representation-finite. Then the following hold:*

(1) If the Auslander algebra of Λ is τ-tilting finite, then so is $T_2(\Lambda)$.

(2) If Λ is simply-connected, then $T_2(\Lambda)$ is τ-tilting finite if and only if it is representation-finite. In particular, the converse of (1) holds.

Proof. Let us first recall an argument in [3, Theorem 1.1]. It was shown that the functor $\Phi : \mathsf{mod}\,T_2(\Lambda) \to \mathsf{mod}(\mathsf{mod}\,\Lambda)$ sending (M, N, f) to $\mathrm{Coker}\,\mathrm{Hom}_\Lambda(-, f)$ is full and dense. Denote by $\mathcal{D}$ the full subcategory of $\mathsf{mod}\,T_2(\Lambda)$ consisting of modules without indecomposable summands of the forms (M, M, id) and $(M, 0, 0)$, where M is an indecomposable module over Λ. Then the restriction of Φ is full and dense (not faithful!), and a morphism σ in $\mathcal{D}$ with $\Phi(\sigma)$ isomorphic is an isomorphism.

We show the assertion (1) holds true. As above, any brick over $T_2(\Lambda)$ lying in $\mathcal{D}$ is sent to some brick in $\mathsf{mod}(\mathsf{mod}\,\Lambda)$ by the functor Φ and the correspondence is objectively injective. Therefore, $T_2(\Lambda)$ inherits the finiteness of bricks from the Auslander algebra of Λ, whence the assertion follows from [4, Theorem 4.2].

To prove the assertion (2), we assume that Λ is simply-connected and $T_2(\Lambda)$ is τ-tilting finite. Then, $T_2(\Lambda)$ does not contain a finite convex subcategory which is concealed of extended Dynkin type. The simple-connectedness of Λ (i.e. $\widetilde{\Lambda} = \Lambda$ in the sense of [8]) implies that $T_2(\Lambda)$ is representation-finite by [8, Theorem 4]. Moreover, we deduce from [3, Theorem 1.1] that the Auslander algebra of Λ is also representation-finite, and so it is τ-tilting finite. □

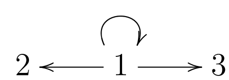
Example 7. Let $\Lambda := K\overrightarrow{A_n}$. Observe that $T_2(\Lambda)$ is the commutative ladder of degree n; see [2, 5, 8]. Then the following are equivalent: (i) $n \leq 4$; (ii) $T_2(\Lambda)$ is representation-finite; (iii) it is τ-tilting finite.

Combining this observation and Theorem 6(1), we recover [6, Corollary 4.8]; that is, the following are equivalent: (i) $n \leq 4$; (ii) the Auslander algebra of Λ is representation-finite; (iii) it is τ-tilting finite.

We give an example which says that the converse of Theorem 6(1) does not necessarily hold. To show this fact, we introduce the second main theorem.

Theorem 8. *Let R be a finite dimensional local K-algebra and put $\Gamma := R \otimes_K \Lambda$. If Λ is silting-discrete, then we have a poset isomorphism $\mathsf{silt}\,\Lambda \to \mathsf{silt}\,\Gamma$. In particular, Γ is also silting-discrete.*

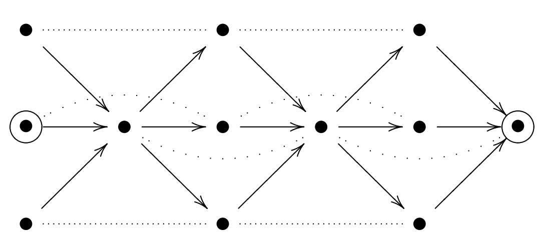
Example 9. Let Λ be the radical-square-zero algebra presented by the quiver:



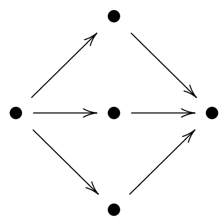
(i) The separated quiver of Λ consists of three connected components; one Dynkin quiver of type D_4 and two isolated points. So, Λ is representation-finite.

(ii) Let us show that $T_2(\Lambda)$ is τ-tilting finite. We consider the algebra A presented by the quiver $2 \longleftarrow 1 \longrightarrow 3$. Since $T_2(A)$ is derived equivalent to the path algebra of Dynkin type E_6 [7], it is seen that $T_2(A)$ is silting-discrete. By Theorem 8, we obtain that $T_2(A) \otimes_K K[x]/(x^2)$ is silting-discrete; in particular, it is τ-tilting finite. As there is an algebra epimorphism $T_2(A) \otimes_K K[x]/(x^2) \to T_2(\Lambda)$, we deduce that the target $T_2(\Lambda)$ is τ-tilting finite.

(iii) However, the Auslander algebra Γ of Λ is not τ-tilting finite. This is deduced by observing the Auslander–Reiten quiver of Λ (it gives a quiver presentation of Γ):



Here, the vertex $\odot$ coincides. Factoring by an ideal, we find the factor algebra Γ_1 of Γ presented by the quiver



with a zero relation; the sum of the three paths of length 2 is zero. Truncating Γ_1 by idempotents, we get the Kronecker algebra, which implies that Γ_1, and so Γ, are τ-tilting infinite.

Finally, we explain the silting-discreteness of triangular matrix algebras.

Theorem 10. *Let Λ, Γ be non-local simply-connected algebras. The following are equivalent.*

(1) $\Lambda \otimes_K \Gamma$ is silting-discrete;
(2) It is a piecewise hereditary algebra of type D_4, E_6 or E_8;
(3) $\Lambda \otimes_K \Gamma$ is derived equivalent to $K\overrightarrow{A_2} \otimes_K \overrightarrow{KA_n}$ ($n \leq 4$).

Corollary 11. *Let Λ be a simply-connected algebra. The following are equivalent.*

(1) $T_2(\Lambda)$ is silting-discrete;
(2) Λ is derived equivalent to $K\overrightarrow{A_n}$ ($n \leq 4$).

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