NUMERICAL TORSION PAIRS AND CANONICAL DECOMPOSITIONS FOR ELEMENTS IN THE GROTHENDIECK GROUP

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ABSTRACT. In the representation theory of finite-dimensional algebras A over a field, it is important to study torsion pairs in the category of finite dimensional A-modules. Baumann–Kamnitzer–Tingley introduced a nice subclass called numerical torsion pairs, which is associated to each element of the Grothendieck group $K_0(\mathsf{proj}\,A)$ of the category of finitely generated projective A-modules, and all functorially finite torsion pairs are realized by numerical torsion pairs. To investigate numerical torsion pairs, we use canonical decompositions of elements in $K_0(\mathsf{proj}\,A)$ introduced by Derksen–Fei. We explain some of our results on the relationship between numerical torsion pairs and canonical decompositions.

1. MOTIVATION

The representation theory of finite-dimensional algebras A over an algebraically closed field K investigates the finite-dimensional modules over A. One of the important problems is to get information on nice subcategories of the category $\mathsf{mod}\,A$ of finite-dimensional A-modules.

A full subcategory $\mathcal{T} \subset \mathsf{mod}\,A$ is called a *torsion class* if it is closed under taking quotients and extensions of modules in $\mathsf{mod}\,A$. It is very difficult to classify all torsion classes, but the subclass called functorially finite torsion classes are well-studied by many authors including [1, 3, 7]. Here, we say that a torsion class $\mathcal{T} \subset \mathsf{mod}\,A$ is *functorially finite* if there exists $M \in \mathsf{mod}\,A$ such that

$$\mathcal{T} = \mathsf{Fac}\,M := \{X \in \mathsf{mod}\,A \mid \text{there exists a surjection } M^{\oplus s} \to X\}.$$

These torsion classes have many nice properties. On the other hand, they are very few among the all torsion classes in general. To study more torsion classes, we deal with *numerical torsion classes* $\mathcal{T}_\theta$, $\overline{\mathcal{T}}_\theta$ associated to each element θ of the Grothendieck group $K_0(\mathsf{proj}\,A)$ of the category of finitely generated projective A-modules. They were introduced by Baumann–Kamnitzer–Tingley [4] (cf. [11]) defined via the *Euler form* and *stability conditions.*

For the study of numerical torsion classes, we can use the *canonical decomposition* $\theta = \bigoplus_{i=1}^{m} \theta_i$ for each element $\theta \in K_0(\mathsf{mod}\,A)$ introduced by Derksen–Fei [8], which comes from decompositions of *2-term complexes* in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ into indecomposable direct summands.

We state some of our results on the relationship between numerical torsion classes and canonical decompositions.

The detailed version of this paper will be submitted for publication elsewhere.

2. Setting

In this proceeding, we assume that K is an algebraically closed field and A is a finite-dimenisonal algebra over K. We write $\mathsf{proj}\,A$ for the category of finitely generated projective A-modules, $P_1, P_2, \dots, P_n$ denote the non-isomorphic indecomposable projective A-modules, and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ means the homotopy category of bounded complexes over $\mathsf{proj}\,A$. Similarly, we set $\mathsf{mod}\,A$ for the category of finite-dimensional A-modules, $S_1, S_2, \dots, S_n$ denote the non-isomorphic simple A-modules, and $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$ means the derived category of bounded complexes over $\mathsf{mod}\,A$. We may assume that there exists a surjection $P_i \to S_i$ for each i. The Grothendieck group of any exact or triangulated category $\mathcal{C}$ is denoted by $K_0(\mathcal{C})$ as usual. We also consider the real Grothendieck group $K_0(\mathcal{C})_{\mathbb{R}} := K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$.

We recall some basic properties on the Grothendieck groups.

Proposition 1. [10] *The following statements hold.*

1. *The Grothendieck groups $K_0(\mathsf{proj}\,A)$ and $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A))$ have $[P_1], [P_2], \dots, [P_n]$ as a $\mathbb{Z}$-basis.*
2. *The Grothendieck groups $K_0(\mathsf{mod}\,A)$ and $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$ have $[S_1], [S_2], \dots, [S_n]$ as a $\mathbb{Z}$-basis.*
3. *For any $i, j \in \{1, 2, \dots, n\}$, we have $\langle [P_i], [S_j] \rangle = \delta_{i,j}$, where $\langle !, ? \rangle \colon K_0(\mathsf{proj}\,A) \times K_0(\mathsf{mod}\,A) \to \mathbb{Z}$ is the Euler form.*

Via the Euler form, each element $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ induces the $\mathbb{R}$-linear form $\theta := \langle \theta, ? \rangle \colon K_0(\mathsf{mod}\,A)_{\mathbb{R}} \to \mathbb{R}$.

3. Numerical torsion pairs

We give the definition of numerical torsion pairs.

Definition 2. [4, Subsection 3.1](cf. [11, Definition 1.1]) Let $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$. We define *numerical torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ by

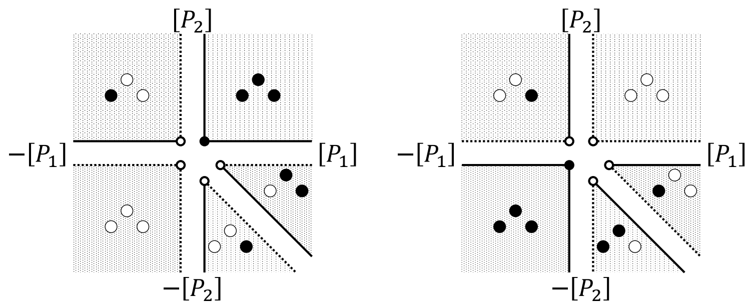
$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(N) \geq 0 \text{ for any quotient module } N \text{ of } M\}, \\
\mathcal{F}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\}, \\
\mathcal{T}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(N) > 0 \text{ for any quotient module } N \neq 0 \text{ of } M\}, \\
\overline{\mathcal{F}}_\theta &:= \{M \in \mathsf{mod}\,A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

By using these, we can define an equivalence relation on $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$.

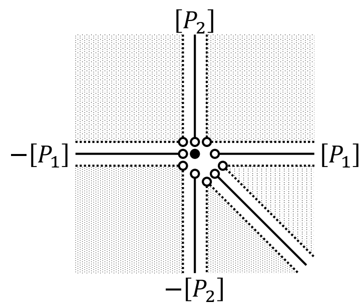
Definition 3. Let $\theta, \theta' \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$. We say that θ and θ' are *TF equivalent* if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'})$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'})$.

We show an example.

Example 4. Let $A = K(1 \to 2)$. Then, the indecomposable A-modules are ${}_{S_2}{}^{P_1}{}_{S_1}$. We can describe torsion(-free) classes by substituting each module in this diagram by $\bullet$ (belongs) or $\circ$ (does not belong). Under this notation, $\overline{\mathcal{T}}_\theta$ and $\overline{\mathcal{F}}_\theta$ are given as follows.



Therefore, $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ has exactly eleven TF equivalence classes.



4. Canonical decompositions

In this section, we deal with canonical decompositions in the Grothendieck group $K_0(\mathsf{proj}\, A)$. We first recall the definition of presentation spaces.

Definition 5. [8, Definition 1.1] Let $\theta \in K_0(\mathsf{proj}\, A)$.

(1) Take $P_+, P_- \in \mathsf{proj}\, A$ (unique up to isomorphisms) such that $\theta = [P_+] - [P_-]$ and $\mathsf{add}\, P_+ \cap \mathsf{add}\, P_- = \{0\}$.
(2) We define the *presentation space* of θ by $\mathsf{Hom}(\theta) = \mathsf{Hom}_A(P_-, P_+)$.
(3) For each $f \in \mathsf{Hom}(\theta)$, we set a 2-term complex $P_f := (P_- \xrightarrow{f} P_+) \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ whose terms except -1st and 0th ones vanish.

Since the presentation space of θ is a K-vector space, it is an affine variety with the Zariski topology.

Then, we can define direct sums in $K_0(\mathsf{proj}\, A)$.

Definition 6. [8, Section 4] Let $\theta_1, \theta_2, \ldots, \theta_m \in K_0(\mathsf{proj}\, A)$. We write $\bigoplus_{i=1}^m \theta_i$ in $K_0(\mathsf{proj}\, A)$ if any general $f \in \mathsf{Hom}(\sum_{i=1}^m \theta_i)$ admits $f_i \in \mathsf{Hom}(\theta_i)$ such that $P_f \cong \bigoplus_{i=1}^m P_{f_i}$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. In this case, we also write $\sum_{i=1}^m \theta_i = \bigoplus_{i=1}^m \theta_i$.

We have the following useful criterion.

Proposition 7. [8, Corollary 4.2, Theorem 4.4] *Let* $\theta_1, \theta_2, \ldots, \theta_m \in K_0(\mathsf{proj}\, A)$. *Then,* $\bigoplus_{i=1}^m \theta_i$ *is equivalent to that, for each* $i \neq j$, *there exist* $(f, f') \in \mathsf{Hom}(\theta_i) \times \mathsf{Hom}(\theta_j)$ *such* *that*

$$\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)}(P_f, P_{f'}[1]) = 0, \quad \mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)}(P_{f'}, P_f[1]) = 0.$$

We also define indecomposable elements in $K_0(\mathsf{proj}\,A)$.

Definition 8. Let $\theta \in K_0(\mathsf{proj}\,A)$. We say that θ is *indecomposable* in $K_0(\mathsf{proj}\,A)$ if, for any general $\mathsf{Hom}(\theta)$, the complex $P_f \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ is indecomposable.

Now, we can define canonical decompositions.

Theorem 9. [12, Theorem 2.7] *Let $\theta \in K_0(\mathsf{proj}\,A)$. Then, θ admits a decomposition $\theta = \bigoplus_{i=1}^m \theta_i$ such that each θ_i is indecomposable in $K_0(\mathsf{proj}\,A)$. It is up to reordering, and we call it the canonical decomposition of θ.*

5. Silting theory

In this section, we recall some basic facts in silting theory.

Definition 10. Let $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ be a 2-term complex. Then, we say that U is *2-term presilting* if $\mathsf{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)}(U, U[1]) = 0$.

2-term presilting complexes have strong relationship with canonical decompositions.

Proposition 11. [12, Lemma 2.16][7, Theorem 6.5] *Let $U = \bigoplus_{i=1}^m U_i$ be a 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ with U_i indecomposable.*

(1) *Any general $f \in \mathsf{Hom}([U])$ satisfies $P_f \cong U$ in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.*
(2) *The canonical decomposition of $[U]$ in $K_0(\mathsf{proj}\,A)$ is $[U] = \bigoplus_{i=1}^m [U_i]$.*

We next consider the relationship between 2-term presilting complexes and TF equivalence classes. From each 2-term presilting complex, we can define torsion pairs. Let U be a 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. We set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^{\perp}H^{-1}(\nu U), \mathsf{Sub}\,H^{-1}(\nu U)), \quad (\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathsf{Fac}\,H^0(U), H^0(U)^{\perp}).$$

Then, we have $\mathcal{T}_U \subset \overline{\mathcal{T}}_U$ and $\mathcal{F}_U \subset \overline{\mathcal{F}}_U$.

Theorem 12. [14, 3, 1] *Let U be a 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.*

(1) *The torsion pairs $(\overline{\mathcal{T}}_U, \mathcal{F}_U), (\mathcal{T}_U, \overline{\mathcal{F}}_U)$ are functorially finite torsion pairs.*
(2) *All functorially finite torsion(-free) classes are obtained in the way of (1).*

The following result claims that each 2-term presilting complex gives a TF equivalence class.

Theorem 13. [15, Proposition 3.3][5, Proposition 3.27][2, Proposition 3.11] *Let $U = \bigoplus_{i=1}^m U_i$ be a 2-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ with U_i indecomposable, and $\eta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$. Then, $\eta \in \sum_{i=1}^m \mathbb{R}_{>0}[U_i]$ is equivalent to $\overline{\mathcal{T}}_\eta = \overline{\mathcal{T}}_U$ and $\overline{\mathcal{F}}_\eta = \overline{\mathcal{F}}_U$. In this case, $\overline{\mathcal{T}}_\eta = \bigcap_{i=1}^m \overline{\mathcal{T}}_{U_i}$ and $\overline{\mathcal{F}}_\eta = \bigcap_{i=1}^m \overline{\mathcal{F}}_{U_i}$ hold.*

6. Main results

Motivated by Theorem 13, we proved the following result in the discussion with Laurent Demonet when we were in Nagoya University.

Theorem 14. *Let* $\theta = \bigoplus_{i=1}^m \theta_i$ *in* $K_0(\mathsf{proj}\, A)$ *and* $\eta \in \sum_{i=1}^m \mathbb{R}_{>0}\theta_i$. *Then, we have* $\overline{\mathcal{T}}_\eta = \bigcap_{i=1}^m \overline{\mathcal{T}}_{\theta_i}$ *and* $\overline{\mathcal{F}}_\eta = \bigcap_{i=1}^m \overline{\mathcal{F}}_{\theta_i}$. *Thus, for any* $i \in \{1, 2, \ldots, m\}$, *we get* $\mathcal{T}_{\theta_i} \subset \mathcal{T}_\eta \subset \overline{\mathcal{T}}_\eta \subset \overline{\mathcal{T}}_{\theta_i}$ *and* $\mathcal{F}_{\theta_i} \subset \mathcal{F}_\eta \subset \overline{\mathcal{F}}_\eta \subset \overline{\mathcal{F}}_{\theta_i}$.

In particular, we can recover the sign-coherence of direct summands of elements in $K_0(\mathsf{proj}\, A)$.

Proposition 15. [12, Lemma 2.10] *Let* $\theta \oplus \theta'$ *in* $K_0(\mathsf{proj}\, A)$, $\theta = \sum_{i=1}^n a_i[P_i]$ *and* $\theta' = \sum_{i=1}^n a'_i[P_i]$. *Then,* $a_i a'_i \geq 0$ *for each* i.

By Theorem 14, if $\theta = \bigoplus_{i=1}^m \theta_i$ is a canonical decomposition in $K_0(\mathsf{proj}\, A)$, then $\sum_{i=1}^m \mathbb{R}_{>0}\theta_i$ is contained in some TF equivalence class. We do not know whether it is always a TF equivalence class. However, we have the following sufficient condition.

Theorem 16. *Assume that one of the following conditions hold:*

(a) *A is a hereditary algebra; or*
(b) *$\theta \oplus \theta$ holds in $K_0(\mathsf{proj}\, A)$ for any $\theta \in K_0(\mathsf{proj}\, A)$.*

If $\theta = \bigoplus_{i=1}^m \theta_i$ *is a canonical decomposition in* $K_0(\mathsf{proj}\, A)$, *then* $\sum_{i=1}^m \mathbb{R}_{>0}\theta_i$ *is a TF equivalence class in* $K_0(\mathsf{proj}\, A)_\mathbb{R}$.

We end this proceeding by explaining the condition (b). If the algebra A satisfies (b), we say that A is *E-tame*. Though it is not easy to check whether the condition (b) holds for a given algebra, Geiss–Labardini-Fragoso–Schröer [9, Theorem 3.2] showed that every representation-finite or tame algebra is E-tame. This depends on the strong results of [6] on 1-parameter families of modules over representation-tame algebras. See also [13].

References

[1] T. Adachi, O. Iyama, I. Reiten, *τ-tilting theory*, Compos. Math. **150**, no. 3 (2014), 415–452.
[2] S. Asai, *The wall-chamber structures of the real Grothendieck groups*, Adv. Math. **381** (2021), 107615.
[3] M. Auslander, S. O. Smalø, *Almost split sequences in subcategories*, J. Algebra **69**, no. 2 (1981), 426–454.
[4] P. Baumann, J. Kamnitzer, P. Tingley, *Affine Mirković-Vilonen polytopes*, Publ. Math. Inst. Hautes Études Sci. **120** (2014), 113–205.
[5] T. Brüstle, D. Smith, H. Treffinger, *Wall and Chamber Structure for finite-dimensional Algebras*, Adv. Math. **354** (2019), 106746.
[6] W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. (3) **56**, Issue 3 (1988), 451–483.
[7] L. Demonet, O. Iyama, G. Jasso, *τ-tilting finite algebras, bricks, and g-vectors*, Int. Math. Res. Not., IMRN 2019, Issue 3, 852–892.
[8] H. Derksen, J. Fei, *General presentations of algebras*, Adv. Math. **278** (2015), 210–237.
[9] C. Geiß, D. Labardini-Fragoso, J. Schröer, *Schemes of modules over gentle algebras and laminations of surfaces*, Sel. Math. New Ser. **28**, 8 (2022), `https://doi.org/10.1007/s00029-021-00710-w`.
[10] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, 119, Cambridge University Press, 1988.

[11] A. D. King, *Moduli of representations of finite dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45**, no. 180 (1994), 515–530.
[12] P.-G. Plamondon, *Generic bases for cluster algebras from the cluster category*, Int. Math. Res. Not., IMRN 2013, no. 10, 2368–2420.
[13] P.-G. Plamondon, T. Yurikusa, with an appendix by B. Keller, *Tame algebras have dense g-vector fans*, Int. Math. Res. Not., https://doi.org/10.1093/imrn/rnab105.
[14] S. O. Smalø, *Torsion theories and tilting modules*, Bull. London Math. Soc. **16**, no. 5 (1984), 518–522.
[15] T. Yurikusa, *Wide subcategories are semistable*, Doc. Math. **23** (2018), 35–47.

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