ON TWO-SIDED HARADA RINGS

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Abstract. In [11] M. Harada studied a left artinian ring R such that every non-small left R-module contains a non-zero injective submodule. (We can see the results also in his lecture note [12, §10.2].) In [16] K. Oshiro called the ring a left H-ring and later in [17] he called it a left Harada ring. Since then many significant results are invented. We can see many results on left Harada rings in [9] and many equivalent conditions in [7, Theorem B]. But results on two-sided Harada rings are few until [1], [2], [4] and [3]. In this paper, we give the structure of two-sided Harada rings

In §1 we give basic definitions including H-epimorphisms, left co-H-sequences and w-co-H-sequences which are induced to characterized two-sided Harada rings. In §2, we study H-epimorphisms and left co-H-sequences In §3, we given important two one-to-one correspondences between the set of all left w-co-H-sequences and the set of all right w-co-H-sequences. In §4, we consider a new concept QF-well-indexed set. In §5, we construct a two-sided Harada ring from a given QF ring uning QF-well-indexed set. In §6, we consider a ring $R(f)$ which is induced from a two-sided Harada ring R. In §7, we show that if R is a two-sided Harada ring R but not QF, then R is isomorphis to a two-sided Harada ring constructed in §5 using a QF ring $R(f)$.

1. Definitions

Let R be a basic artinian ring. A ring R is called a *left Harada ring* or a *left H-ring* if, for any primitive idempotent e of R, there exists a primitive idempotent f_e of R with $E(T({}_RRe)) \cong {}_RRf_e/S_{n_e}({}_RRf_e)$ for some $n_e \in \mathbb{N}$.

By, for instance, [7, Theorem B (5),(6),(14) and the proof of (6) $\Rightarrow$ (5)], the following are equivalent:

- (a) R is a left Harada ring.
- (b) There exist a basic set $\{ e_{i,j} \}_{i=1,j=1}^{m\ \ n(i)}$ of orthogonal primitive idempotents of R and a set $\{ f_i \}_{i=1}^{m}$ of primitive idempotents of R such that $E(T({}_RRe_{i,j})) \cong {}_RRf_i/S_{j-1}({}_RRf_i)$ for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n(i)$.
- (c) There exists a basic set $\{ e_{i,j} \}_{i=1,j=1}^{m\ \ n(i)}$ of orthogonal primitive idempotents of R such that $e_{i,1}R_R$ is injective and $e_{i,j}R_R \cong e_{i,1}J_R^{j-1}$ for each $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n(i)$.

We may consider the sets

$$\{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$$

in $(b), (c)$ coincide and call it a *well-indexed set of left Harada ring* or a *left well-indexed set*.

The detailed version of this paper will be submitted for publication elsewhere.

Further, for primitive idempotents e, f of R, we call

$$(eR,\ Rf)$$

is an *i-pair* if both $S(eR_R) \cong T(fR_R)$ and $S(_RRf) \cong T(_RRe)$ hold. And, since $\{e_{i,1}R\}_{i=1}^{m}$ is a basic set of indecomposable projective injective right R-modules, for each $i = 1, 2, \ldots, m$, there exists $e_{\sigma(i),\rho(i)} \in \{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$ such that $(e_{i,1}R,\ Re_{\sigma(i),\rho(i)})$ is an i-pair by [10, Theorem 3.1], where $\sigma,\ \rho : \{1, 2, \ldots, m\} \to \mathbb{N}$ are mappings.

Unless otherwise stated, throughout this paper, we let R be an indecomposable basic two-sided Harada ring, let $\{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$ be its well-indexed set of left Harada ring, let σ, ρ be mappings above, and, for each $i = 1, 2, \ldots, m$ and each $j = 2, 3, \ldots, n(i)$, let

$$\theta_{i,j} : e_{i,j}R_R \to e_{i,j-1}J_R$$

be an R-isomorphism.

Let R be an artinian ring, let $\{e_i\}_{i=1}^{n}$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^{k} \subseteq \{e_i\}_{i=1}^{n}$. A sequence $f_1R,\ f_2R,\ \ldots,\ f_kR$ is called a *right co-H-sequence* of R if the following (CHS1), (CHS2), (CHS3) hold.

- (CHS1) For each $i = 1, 2, \ldots, k-1$, there exists an R-isomorphism $\xi_i : f_iR_R \to f_{i+1}J_R$.
- (CHS2) The last term f_kR_R is injective.
- (CHS3) $f_1R,\ f_2R,\ \ldots,\ f_kR$ is the longest sequence among the sequences which satisfy both (CHS1) and (CHS2), i.e., there does not exist an R-isomorphism: $fR_R \to f_1J_R$, where $f \in \{e_i\}_{i=1}^{n}$.

Similarly, we define a *left co-H-sequence* $Rf_1,\ Rf_2,\ \ldots,\ Rf_k$ of R.

Obviously, for each $i = 1, 2, \ldots, m$

$$e_{i,n(i)}R_R,\ e_{i,n(i)-1}R_R,\ \ldots,\ e_{i,1}R_R$$

is a right co-H-sequence of R. And, for an artinian ring R', it is a left Harada ring if and only if there exists a basic set $\{e_{i,j}\}_{i=1,j=1}^{m\ \ n(i)}$ of orthogonal primitive idempotents of R' such that $e_{i,n(i)}R',\ e_{i,n(i)-1}R',\ \ldots,\ e_{i,1}R'$ is a right co-H-sequence of R' for all $i = 1, 2, \ldots, m$.

From the definition of a left Harada ring, the following lemma holds:

Lemma 1. *For a left Harada ring R' and primitive idempotents $f_1,\ f_2,\ \ldots,\ f_k$ of R', the following are equivalent.*

- (a) $f_1R',\ f_2R',\ \ldots,\ f_kR'$ *is a right co-H-sequence.*
- (b) $f_1R',\ f_2R',\ \ldots,\ f_kR'$ *satisfies* (CHS1) *and the following* (CHS3$'$)*:*
  - (CHS3$'$) $f_1R',\ f_2R',\ \ldots,\ f_kR'$ *is the longest sequence among sequences which satisfy* (CHS1).

Let $\{e_i\}_{i=1}^{n}$ be a complete set of orthogonal primitive idempotents of R and let $\{f_i\}_{i=1}^{j+1} \subseteq \{e_i\}_{i=1}^{n}$, where $f_1, f_2, \ldots, f_{j+1}$ are mutually distinct. Then we call $\varphi : f_1R_R \to f_2J_R$ (resp.

${}_RRf_1 \to {}_RJf_2$) a *right* (resp. *left*) *H-epimorphism* if φ is a non-zero R-epimorphism with $J \cdot \operatorname{Ker}\varphi = 0$ (resp. $\operatorname{Ker}\varphi \cdot J = 0$). And we call $\varphi : f_1R_R \to f_{j+1}J_R^j$ (resp. ${}_RRf_1 \to {}_RJ^jf_{j+1}$) a *right* (resp. *left*) *weak H-epimorphism* (or simply a *right* (resp. *left*) *w-H-epimorphism*) if there exist right (resp. left) H-epimorphisms $\varphi_i : f_iR_R \to f_{i+1}J_R$ ($i = 1, 2, \ldots, j$) with $\varphi = \varphi_j\varphi_{j-1}\cdots\varphi_1$ (resp. $\varphi_i : {}_RRf_i \to {}_RJf_{i+1}$ ($i = 1, 2, \ldots, j$) with $\varphi = \varphi_1\varphi_2\cdots\varphi_j$).

We call a sequence f_1R, f_2R, $\ldots$, f_kR a *right weak co-H-sequence* (or simply a *right w-co-H-sequence*) if the following (WCHS1), (WCHS2) hold.

(WCHS1) For any $i = 1, 2, \ldots, k-1$, there exists a right H-epimorphism ξ_i : $f_iR_R \to f_{i+1}J_R$.

(WCHS2) There exists neither a right H-epimorphism $\xi : fR_R \to f_1J_R$ nor a right H-epimorphism $\xi' : f_kR_R \to f'J_R$ for any $f,\ f' \in \{e_i\}_{i=1}^n - \{\,f_i\,\}_{i=1}^k$, i.e., f_1R, f_2R, $\ldots$, f_kR is the longest sequence in the set of all sequences which consist of distinct terms and satisfy the condition (WCHS1).

Further a right w-co-H-sequence f_1R, f_2R, $\ldots$, f_kR is called a *right cyclic weak co-H-sequence* if there exists a right H-epimorphism $\xi_k : f_kR_R \to f_1J_R$.

Similarly, we define a *left* (*cyclic*) *weak co-H-sequence* Rf_1, Rf_2, $\ldots$, Rf_k .

We call an artinian ring R a *QF ring* if R is injective as a left (or right) R-module.

Let Q be an indecomposable basic QF ring. Then we call $\{\,f'_{i,s}\,\}_{i=1,s=1}^{m'\ \delta'_i}$ a *left QF-well-indexed set* of Q if $\{\,f'_{i,s}\,\}_{i=1,s=1}^{m'\ \delta'_i}$ is a complete set of orthogonal primitive idempotents of Q which satisfies the following two conditions:

(QFWI 1) $Qf'_{i,1}$, $Qf'_{i,2}$, $\ldots$, Qf'_{i,δ'_i} is a left w-co-H-sequence for any $i = 1, 2, \ldots, m'$.

(QFWI 2) If $\delta'_i \geq 2$, then ($f'_{i,s}Q$, $Qf'_{i,s}$) is an i-pair for any $s = 1, 2, \ldots, \delta'_i$.

We call an artinian ring R is a *Nakayama ring* if both ${}_RRe$ and eR_R are uniserial for any primitive idempotent e of R.

For $a \in R$, we write the left (resp. right) multiplication map by a

$$(a)_L \quad (\text{resp. } (a)_R\,)\,.$$

And, for primitive idempotents e, f and g, we use the following terminologies.

- If both $S({}_{eRe}eRf)$ and $S(eRf_{fRf})$ are simple, we call (eR, Rf) is a *colocal pair* following [13] and [15]. And then $S({}_{eRe}eRf) = S(eRf_{fRf})$ holds. We abbreviate it to

$$S(eRf)\,.$$

- We put

$$R(e) \overset{put}{:=} eRe\,.$$

2. H-epimorphisms and left co-H-sequences of two-sided Harada rings

We characterize left (right) H-epimorphisms.

Theorem 2.

(I) Suppose that $\zeta : {}_R Re_{i,j} \to {}_R Je_{k,l}$ is a left H-epimorphism. And, if ${}_R Re_{i,j}$ is injective, we let $(e_{p,1}R,\ Re_{i,j})$ be an i-pair. Then the following hold.

(1) (*i*) Suppose that $e_{k,l}R_R$ is injective, i.e., $l = 1$. Then $j = n(i)$, i.e., $\zeta : {}_R Re_{i,n(i)} \to {}_R Je_{k,1}$.

(*ii*) Suppose that $e_{k,l}R_R$ is not injective, i.e., $l \neq 1$. Then $(k,l) = (i, j+1)$ $(\, j < n(i)\,)$, i.e., $\zeta : {}_R Re_{i,j} \to {}_R Je_{i,j+1}$.

(2) (*i*) $\operatorname{Ker} \zeta = S(R_R)\, e_{i,j} =$

$$\begin{cases} \oplus_{q=1}^{n(p)} S(e_{p,q}R_R) = S_{n(p)}({}_R Re_{i,j}) \neq 0 \text{ and it is uniserial as a left } R\text{-module} & (\text{if } {}_R Re_{i,j} \text{ is injective}) \\ 0 & (\text{if } {}_R Re_{i,j} \text{ is not injective}) \end{cases}$$

(*ii*) If ${}_R Re_{i,j}$ is injective, then, for each $q = 1, 2, \dots, n(p)$,
$S(e_{p,q}R_R) = S(e_{p,q}R_R)\, e_{i,j} = S(e_{p,q}Re_{i,j})$.

(II) Suppose that $\xi : e_{i,j}R_R \to e_{k,l}J_R$ is a right H-epimorphism. And, if $e_{i,j}R_R$ is injective, we put $I_i \overset{put}{:=} \{\ (p,q) \mid S({}_R Re_{p,q}) \cong T({}_R Re_{i,1})\ \}$ and let n' be the number of elements in I_i. Then the following hold.

(1) (*i*) Suppose that $e_{i,j}R_R$ is injective, i.e., $j = 1$. Then $l = n(k)$, i.e., $\xi : e_{i,1}R_R \to e_{k,n(k)}J_R$.

(*ii*) Suppose that $e_{i,j}R_R$ is not injective, i.e., $j \geq 2$. Then $(k,l) = (i, j-1)$ $(\, l < n(k)\,)$, i.e., $\xi : e_{i,j}R_R \to e_{i,j-1}J_R$.

(2) (*i*) $\operatorname{Ker} \xi = e_{i,j}\, S({}_R R) =$

$$\begin{cases} \oplus_{(p,q)\in I_i} e_{i,1}\, S({}_R Re_{p,q}) = S_{n'}(e_{i,1}R_R) \neq 0 \text{ and it is uniserial as a right } R\text{-module} & (\text{if } j = 1) \\ 0 & (\text{if } j \neq 1) \end{cases}$$

(*ii*) If $e_{i,j}R_R$ is injective, i.e., $j = 1$, then, for each $(p,q) \in I_i$,
$S({}_R Re_{p,q}) = e_{i,1}\, S({}_R Re_{p,q}) = S(e_{i,1}Re_{p,q})$.

By the definition of a well-indexed set $\{\, e_{i,j}\,\}_{i=1,j=1}^{m\ n(i)}$ of left Harada ring,

$$e_{i,n(i)}R,\ e_{i,n(i)-1}R, \dots, e_{i,1}R \quad (\, i = 1, 2, \dots, m\,)$$

are right co-H-sequences of R. And, from Theorem 2, we obtain the following characterization left co-H-sequences of R using the same set $\{\, e_{i,j}\,\}_{i=1,j=1}^{m\ n(i)}$.

Corollary 3. Every left co-H-sequence of R is of the form

$$Re_{i_1,s},\ Re_{i_1,s+1}, \dots,\ Re_{i_1,n(i_1)},\ Re_{i_2,1},\ Re_{i_2,2},\ \dots,\ Re_{i_2,n(i_2)},\ Re_{i_3,1},\ \dots\ , Re_{i_u,t}\ ,$$

where $1 \le i_1, i_2, \ldots, i_u \le m$, $1 \le s \le n(i_1)$ and $1 \le t \le n(i_u)$.

Example 4. Let R be a basic indecomposable Nakayama ring with a complete set $\{ g_i \}_{i=1}^{7}$ of orthogonal primitive idempotents which satisfies

(*i*) $T(g_i J_R) \cong T(g_{i+1} R_R)$ for any $i = 1, 2, \ldots, 6$, and

(*ii*) $c(g_1 R_R) = 10, \quad c(g_2 R_R) = 9,$
$c(g_3 R_R) = 10, \quad c(g_4 R_R) = 9,$
$c(g_5 R_R) = 11, \quad c(g_6 R_R) = 10, \quad c(g_7 R_R) = 9,$

where $c(M)$ means the composition length of an R-module M.

We put

$$e_{1,1} \overset{put}{:=} g_1, \quad e_{1,2} \overset{put}{:=} g_2, \quad e_{2,1} \overset{put}{:=} g_3, \quad e_{2,2} \overset{put}{:=} g_4, \quad e_{3,1} \overset{put}{:=} g_5, \quad e_{3,2} \overset{put}{:=} g_6, \quad e_{3,3} \overset{put}{:=} g_7.$$

And $\{ e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{2,2},\ e_{3,1},\ e_{3,2},\ e_{3,3} \}$ is a left well-indexed set of R and

$$(e_{1,1}R,\ Re_{2,1}), \quad (e_{2,1}R,\ Re_{3,1}), \quad (e_{3,1}R,\ Re_{1,1})$$

are i-pairs and

$$Re_{1,2},\ Re_{2,1}$$
$$Re_{2,2},\ Re_{3,1}$$
$$Re_{3,2},\ Re_{3,3},\ Re_{1,1}$$

are left co-H-sequences.

3. Two one-to-one correspondeces between $\mathbf{S}_L$ and $\mathbf{S}_R$.

In the following lemma, we give the form of left (right) weak co-H-sequences.

Lemma 5.

(I) (*i*) *Every left non-cyclic w-co-H-sequence is of the form*

$$Rf'_{1,1}, Rf'_{1,2}, \cdots, Rf'_{1,n_1}, Rf'_{2,1}, Rf'_{2,2}, \cdots, Rf'_{2,n_2}, Rf'_{3,1}, Rf'_{3,2}, \cdots\cdots$$
$$\cdots\cdots, Rf'_{k-1,n_{k-1}}, Rf'_{k,1}, Rf'_{k,2}, \cdots, Rf'_{k,n_k},$$

where we let $Rf'_{i,1}, Rf'_{i,2}, \cdots, Rf'_{i,n_i}$ *be a left co-H-sequence for each* $i = 1, 2, \ldots, k$.

(*ii*) *And we may consider that every left cyclic w-co-H-sequence is also of the same form by renumbering the indexes if necessary.*

(II) (*i*) *We may assume that every right non-cyclic w-co-H-sequence is of the form*

$$e_{i_k,n(i_k)}R,\ e_{i_k,n(i_k)-1}R,\ \cdots,\ e_{i_k,1}R,\ e_{i_{k-1},n(i_{k-1})}R,\ e_{i_{k-1},n(i_{k-1})-1}R,\ \cdots$$
$$e_{i_{k-1},1}R,\ e_{i_{k-2},n(i_{k-2})}R,\ \cdots\cdots,\ e_{i_2,1}R,\ e_{i_1,n(i_1)}R,\ e_{i_1,n(i_1)-1}R,\ \cdots,\ e_{i_1,1}R,$$

where $\{ i_1,\ i_2,\ \ldots,\ i_k \} \subseteq \{ 1, 2, \ldots, m \}$.

(*ii*) *And we may consider that every right cyclic w-co-H-sequence is also of the same form by renumbering the indexes if necessary.*

By Lemma 5 (II), we may assume that there exist integers $\alpha_1, \alpha_2, \ldots, \alpha_{m'}$ and $\beta_1, \beta_2, \ldots, \beta_{m'}$ which satisfy the following $(i), (ii)$.

(i) $\alpha_1 = 1,\ 1 \le \beta_1 < \beta_2 < \cdots < \beta_{m'} = m$ and $\alpha_i = \beta_{i-1} + 1$ for any $i = 2, 3, \ldots, m'$.

(ii) For each $i = 1, 2, \ldots, m'$,

$$(R\text{--}i\,)\quad \begin{array}{l} e_{\beta_i,n(\beta_i)}R,\ e_{\beta_i,n(\beta_i)-1}R,\ \ldots\ ,\ e_{\beta_i,1}R,\ e_{\beta_i-1,n(\beta_i-1)}R,\ e_{\beta_i-1,n(\beta_i-1)-1}R,\ \ldots \\ \cdots\ ,\ e_{\beta_i-1,1}R,\ e_{\beta_i-2,n(\beta_i-2)}R,\ e_{\beta_i-2,n(\beta_i-2)-1}R,\ \cdots\cdots\ ,\ e_{\alpha_i+1,1}R, \\ e_{\alpha_i,n(\alpha_i)}R,\ e_{\alpha_i,n(\alpha_i)-1}R,\ \cdots\ ,\ e_{\alpha_i,1}R \end{array}$$

is a right w-co-H-sequence.

And, for each $i = 1, 2, \ldots, m'$, we also consider another sequence

$$(L\text{--}i\,)\quad \begin{array}{l} Re_{\alpha_i,1},\ Re_{\alpha_i,2},\ \ldots\ ,\ Re_{\alpha_i,n(\alpha_i)},\ Re_{\alpha_i+1,1},\ Re_{\alpha_i+1,2},\ \ldots,\ Re_{\alpha_i+1,n(\alpha_i+1)}, \\ Re_{\alpha_i+2,1},\ Re_{\alpha_i+2,2},\ \cdots\cdots\ ,\ Re_{\beta_i-1,n(\beta_i-1)},\ Re_{\beta_i,1},\ Re_{\beta_i,2},\ \cdots\ ,\ Re_{\beta_i,n(\beta_i)} \end{array}$$

of left R-modules. Further, we put

$$\mathbf{S}_R \overset{put}{:=} \{\, (R\text{--}i\,) \,\}_{i=1}^{m'} \quad \text{and} \quad \mathbf{S}_L \overset{put}{:=} \{\, (L\text{--}i\,) \,\}_{i=1}^{m'}.$$

Of course, $\mathbf{S}_R$ is the set of all right w-co-H-sequences.

Throughout this paper, we use the notations $\alpha_i,\ \beta_i,\ (R\text{--}i\,),\ (L\text{--}i\,),\ \mathbf{S}_L$ and $\mathbf{S}_R$.

In this section, we give two one-to-one correspondences between $\mathbf{S}_L$ and $\mathbf{S}_R$. The first one is as follows.

Theorem 6.

(1) $\mathbf{S}_L$ is the set of all left w-co-H-sequences.

(2) We can define a bijection

$$\Phi : \mathbf{S}_L \to \mathbf{S}_R$$

by

$$\Phi(\,(L\text{--}i\,)\,) = (R\text{--}i\,)$$

for any $i = 1, 2, \ldots, m'$.

(3) Φ satisfy the following two properties.

(i) Φ preserve the length of a sequence.

(ii) Φ preserve the property that it is cyclic (or not cyclic).

Now we give a key lemma in this paper.

Lemma 7.

(I) *Let*

$$Rf_1,\ Rf_2,\ \ldots\ ,\ Rf_{n_l}$$

be a left co-H-sequence and let $\zeta : {}_RRf_0 \to {}_RJf_1$ be a left H-epimorphism and let $(e_{k,1}R,\ Rf_0)$ and $(e_{l,1}R,\ Rf_{n_l})$ be i-pairs, i.e., $f_0 = e_{\sigma(k),\rho(k)}$ and $f_{n_l} = e_{\sigma(l),\rho(l)}$. Suppose that $f_0 \neq f_{n_l}$, i.e., $k \neq l$. Then there exists a right H-epimorphism $\xi : e_{l,1}R_R \to e_{k,n(k)}J_R$.

(II) *Let $\xi : e_{l,1}R_R \to e_{k,n(k)}J_R$ be a right H-epimorphism and let*

$$Rf_1, Rf_2, \ldots, Rf_{n_l} = Re_{\sigma(l),\rho(l)}$$

be a left co-H-sequence. Suppose that $k \neq l$. Then there exists a left H-epimorphism $\zeta : {}_RRe_{\sigma(k),\rho(k)} \to {}_RJf_1$.

Remark. *In Corollary 10, we will show that there exist ξ in (I) and ζ in (II) even if $k = l$.*

Now, for each $i = 1, 2, \ldots, m'$, we abbreviate

$e_{\alpha_i,1},\ e_{\alpha_i,2},\ \cdots,\ e_{\alpha_i,n(\alpha_i)},\ e_{\alpha_i+1,1},\ e_{\alpha_i+1,2},\ \ldots,\ e_{\alpha_i+1,n(\alpha_i+1)},\ e_{\alpha_i+2,1},\ e_{\alpha_i+2,2},\ \cdots\cdots,e_{\beta_i,n(\beta_i)}$

to

$$f_{i,1},\ f_{i,2},\ \cdots,\ f_{i,\gamma_i}\,.$$

Then a left w-co-H-sequence $(L\!-\!i\,)$ is written by

$$Rf_{i,1},\ Rf_{i,2},\ \ldots,\ Rf_{i,\gamma_i}$$

and a right w-co-H-sequence $(R\!-\!i\,)$ is written by

$$f_{i,\gamma_i}R,\ f_{i,\gamma_i-1}R,\ \ldots,\ f_{i,1}R\,.$$

It is obvious that a right co-H-sequence $(R\!-\!i\,)$ contains $\beta_i-\alpha_i+1=\beta_i-\beta_{i-1}$ injective right R-modules

$$e_{\alpha_i,1}R,\ e_{\alpha_i+1,1}R,\ \ldots,\ e_{\beta_i,1}R\,,$$

where we let $\beta_0 = 0$. Now we assume that a left co-H-sequence $(L\!-\!i\,)$ contains δ_i injective left R-modules

$$Rf_{i,p_i(1)},\ Rf_{i,p_i(2)},\ \ldots,\ Rf_{i,p_i(\delta_i)}\,,$$

where $(\,1\le\,)\ p_i(1)<p_i(2)<\cdots<p_i(\delta_i)\ (\,\le\gamma_i\,)$. Further we let

$$(e_{q_i(j),1}R,\ Rf_{i,p_i(j)})$$

be an i-pair for any $j = 1, 2, \ldots, \delta_i$.

Throughout this paper, we use these notations.

We note that $p_i(\delta_i) = \gamma_i$ does not hold necessarily. But the following Lemma 8 (1) holds. And in Lemma 8 (2), we consider the case that $p_i(\delta_i) \neq \gamma_i$, i.e., $p_i(\delta_i) < \gamma_i$.

Lemma 8.

(1) *Suppose that $(L\!-\!i\,)$ is not cyclic. Then $p_i(\delta_i) = \gamma_i$, i.e., ${}_RRf_{i,\gamma_i}$ is injective.*

(2) *Suppose that $p_i(\delta_i) < \gamma_i$. Then the following hold.*

  (*i*) *$Rf_{i,p_i(\delta_i)+1},\ Rf_{i,p_i(\delta_i)+2},\ \ldots,\ Rf_{i,\gamma_i},\ Rf_{i,1},\ Rf_{i,2},\ \ldots,\ Rf_{i,p_i(1)}$ is a left co-H-sequence.*

  (*ii*) *There exists a right H-epimorphism $\xi' : e_{q_i(1),1}R_R \to e_{q_i(\delta_i),n(q_i(\delta_i))}J_R$.*

Now we give new sequences $[L\!-\!i\,]$ and $[R\!-\!i\,]$ as follows.

Lemma 9. *For $i = 1, 2, \ldots, m'$, we consider the following two sequences:*

$[L\!-\!i\,]$ $Re_{q_i(1),1},\ Re_{q_i(1),2},\ \ldots,\ Re_{q_i(1),n(q_i(1))},\ Re_{q_i(2),1},\ Re_{q_i(2),2},\ \ldots,\ Re_{q_i(2),n(q_i(2))},$ $Re_{q_i(3),1},\ Re_{q_i(3),2},\ \ldots,\ Re_{q_i(\delta_i-1),n(q_i(\delta_i-1))},\ Re_{q_i(\delta_i),1},\ Re_{q_i(\delta_i),2},\ \ldots$ $\ldots,\ Re_{q_i(\delta_i),n(q_i(\delta_i))}\,.$

$$[R\!-\!i\,]\quad \begin{array}{l} e_{q_i(\delta_i),n(q_i(\delta_i))}R,\ e_{q_i(\delta_i),n(q_i(\delta_i))-1}R,\ \ldots,\ e_{q_i(\delta_i),1}R,\ e_{q_i(\delta_i-1),n(q_i(\delta_i-1))}R, \\ e_{q_i(\delta_i-1),n(q_i(\delta_i-1))-1}R,\ \ldots,\ e_{q_i(\delta_i-1),1}R,\ e_{q_i(\delta_i-2),n(q_i(\delta_i-2))}R,\ e_{q_i(\delta_i-2),n(q_i(\delta_i-2))-1}R, \\ \ldots\ldots,\ e_{q_i(2),1}R,\ e_{q_i(1),n(q_i(1))}R,\ e_{q_i(1),n(q_i(1))-1}R,\ \ldots,\ e_{q_i(1),1}R\,. \end{array}$$

Then the following hold.

(1) (*i*) $[L\!-\!i\,]$ *is a left w-co-H-sequence, i.e.,* $[L\!-\!i\,]\in \mathbf{S}_L$.
(*ii*) $[R\!-\!i\,]$ *is a right w-co-H-sequence, i.e.,* $[R\!-\!i\,]\in \mathbf{S}_R$.

(2) *The following are equivalent.*
(*a*) $(L\!-\!i\,)$ *is cyclic.*
(*b*) $(R\!-\!i\,)$ *is cyclic.*
(*c*) $[L\!-\!i\,]$ *is cyclic.*
(*d*) $[R\!-\!i\,]$ *is cyclic.*

The following Corollary complements the statement of Lemma 7.

Corollary 10.

(I) *Suppose that*

$$Rf_1,\ Rf_2,\ \ldots\ ,\ Rf_{n'}$$

is a left co-H-sequence and it is cyclic as a left w-co-H-sequence. Let $(e_{k,1}R,\ Rf_{n'})$ *be an i-pair. Then the right co-H-sequence*

$$e_{k,n(k)}R,\ e_{k,n(k)-1}R,\ \ldots,\ e_{k,1}R$$

is cyclic as a right w-co-H-sequence.

(II) *Suppose that the right co-H-sequence*

$$e_{i,n(i)}R,\ e_{i,n(i)-1}R,\ \ldots,\ e_{i,1}R$$

is cyclic as a right w-co-H-sequence. Then a left co-H-sequence with the last term $Re_{\sigma(i),\rho(i)}$ *is also cyclic as a left w-co-H-sequence.*

Now we give the second one-to-one correspondence between $\mathbf{S}_L$ and $\mathbf{S}_R$.

Theorem 11. *A bijection*

$$\Psi : \mathbf{S}_L \to \mathbf{S}_R$$

is defined by

$$\Psi(\,(L\!-\!i\,)\,) = [R\!-\!i\,]$$

and the following hold.

(*i*) Ψ *preserve the number of injective modules in a w-co-H-sequence.*
(*ii*) Ψ *preserve the property that it is cyclic (or not cyclic).*

We define a bijection

$$\psi : \{\,1, 2, \ldots, m'\,\} \to \{\,1, 2, \ldots, m'\,\}$$

by

$$(R\!-\!\psi(i))\ =\ [R\!-\!i\,],\quad \text{i.e.,}\quad \Psi(\,(L\!-\!i\,)\,)\ =\ (\,R\!-\!\psi(i)\,)\,.$$

Then we note that

$$(f_{\psi(i),1}R,\, Rf_{i,p_i(1)})$$

is an i-pair for any $i = 1, 2, \ldots, m'$ by the definition.

And, for $i = 1, 2, \ldots, m'$, we put

$$f_i \overset{put}{:=} \textstyle\sum_{j=1}^{\gamma_i} f_{i,j} \quad \text{and} \quad R_i \overset{put}{:=} f_i R f_i \, .$$

Throughout this paper, we let ψ mean this bijection and use the notations f_i and R_i.

In the following theorem, we consider the case that $(L{-}i\,)$ is cyclic.

Theorem 12. *Suppose that $(L{-}i\,)$ is cyclic. Then the following hold.*

(1) *R_i is a Nakayama ring.*

(2) $(1 - f_i)Rf_i = f_iR(1 - f_i) = 0\,.$

(3) $\{ f_{i,j} \}_{j=1}^{\gamma_i} = \{ e_{q_i(k),l} \}_{k=1,l=1}^{\delta_i \;\; n(q_i(k))}$, *i.e.,* $\psi(i) = i$, *i.e.,* $\Psi(\,(L{-}i\,)\,) = \Phi(\,(L{-}i\,)\,)\,.$

In the following lemma, we characterize δ_i.

Lemma 13. $\delta_i \;=\; \beta_{\psi(i)} - \alpha_{\psi(i)} + 1\,.$

In the following theorem, $\{ q_i(\,j\,) \}_{i=1,j=1}^{m\;\;\delta_i}$ is, i.e., i-pairs in R are, characterized in (1) and we give a condition to be cyclic for a w-co-H-sequence in (2).

Theorem 14.

(1) (*i*) $q_i(\,\beta_{\psi(i)} - q_i(1) + 1\,) \;=\; \beta_{\psi(i)}\,.$

(*ii*) $q_i(\,\beta_{\psi(i)} - q_i(1) + 2\,) \;=\; \alpha_{\psi(i)}\,.$

(*iii*) $q_i(\,j{+}1\,) \;=\; q_i(\,j\,){+}1$ *holds for any* $j \in \{\,1, 2, \ldots, \delta_i{-}1\,\}{-}\{\,\beta_{\psi(i)}{-}q_i(1){+}1\,\}\,.$

(2) *Suppose that $q_i(1) \;>\; \alpha_{\psi(i)}$. Then $(R{-}\psi(i))$ is cyclic (i.e., $(L{-}i\,)$ is cyclic), $\psi(i) = i$ and R_i is a Nakayama ring.*

Remark 15. Let R be an indecomposable ring. We suppose that R is not a Nakayama ring. Then R_i is also not a Nakayama ring for all $i = 1, 2, \ldots, m$ by Theorem 12 (2). So

$$q_i(1) \;=\; \alpha_{\psi(i)}$$

holds for any $i = 1, 2, \ldots, m$ from Theorem 14 (2). And further

$$q_i(j) \;=\; \alpha_{\psi(i)} + j - 1$$

holds for any $i = 1, 2, \ldots, m'$ and any $j = 1, 2, \ldots, \delta_i = \beta_{\psi(i)} - \alpha_{\psi(i)} + 1$ by Theorem 14 (1)(*iii*).

Example 16. Let $\tilde{R}$ be a *QF* ring with a complete set $\{\tilde{e}_1, \tilde{e}_2\}$ of orthogonal primitive idempotents. And we put $Q_i \overset{put}{:=} \tilde{e}_i\tilde{R}\tilde{e}_i$ $(i = 1, 2)$, $A \overset{put}{:=} \tilde{e}_1\tilde{R}\tilde{e}_2$ and $B \overset{put}{:=} \tilde{e}_2\tilde{R}\tilde{e}_1$.

(1) Suppose that $(\tilde{e}_1\tilde{R},\ \tilde{R}\tilde{e}_2)$ and $(\tilde{e}_2\tilde{R}, \tilde{R}\tilde{e}_1)$ are i-pairs. (Then we note that $A \neq 0$ and $B \neq 0$.) We consider

$$R \overset{put}{:=} \begin{pmatrix} Q_1 & Q_1 & Q_1 & A & \overline{A} \\ J(Q_1) & Q_1 & Q_1 & A & \overline{A} \\ J(Q_1) & J(Q_1) & Q_1 & A & A \\ B & B & B & Q_2 & Q_2 \\ B & B & B & J(Q_2) & Q_2 \end{pmatrix},$$

where $J(Q_i)$ means the Jacobson radical of Q_i for $i = 1, 2$, we put $\overline{A} \overset{put}{:=} A/S(A)$ and, for each $j = 1, 2, \ldots, 5$, let e_j be the j-th matrix unit. Then R is a two-sided Harada ring as follows.

For instance, we put

$$e_{1,1} = e_1,\ \ e_{1,2} = e_2,\ \ e_{2,1} = e_3,\ \ e_{3,1} = e_4,\ \ e_{3,2} = e_5 .$$

Then $\{ e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{3,1},\ e_{3,2} \}$ is a left well-indexed set and

$$\alpha_1 = 1,\ \ \beta_1 = 2,\ \ \alpha_2 = 3,\ \ \beta_2 = 3 ,$$

i.e.,

$(R{-}1)$ $e_{2,1}R,\ e_{1,2}R,\ e_{1,1}R,$
$(R{-}2)$ $e_{3,2}R,\ e_{3,1}R$

are right w-co-H-sequences. So

$(L{-}1)$ $Re_{1,1},\ Re_{1,2},\ Re_{2,1},$
$(L{-}2)$ $Re_{3,1},\ Re_{3,2}$

are left w-co-H-sequences. We put

$$f_{1,1} \overset{put}{:=} e_{1,1},\ \ f_{1,2} \overset{put}{:=} e_{1,2},\ \ f_{1,3} \overset{put}{:=} e_{2,1},\ \ f_{2,1} \overset{put}{:=} e_{3,1},\ \ f_{2,2} \overset{put}{:=} e_{3,2} .$$

Then

$$\delta_1 = 1,\ \ \delta_2 = 2,\ \ \ p_1(1) = 3,\ \ p_2(1) = 1,\ \ p_2(2) = 2 \ \ \text{and}$$

$$q_1(1) = 3,\ \ q_2(1) = 1,\ \ q_2(2) = 2 ,$$

i.e.,

$$(e_{3,1}R,\ Rf_{1,3}),\ \ (e_{1,1}R,\ Rf_{2,1}),\ \ (e_{2,1}R,\ Rf_{2,2})$$

are i-pairs. So

$$\Psi(\,(L{-}1)\,) = [R{-}1] = (R{-}2) \ \ \text{and}\ \ \Psi(\,(L{-}2)\,) = [R{-}2] = (R{-}1) .$$

Therefore

$$\psi(1) = 2,\ \ \psi(2) = 1 \ \ \text{and}\ \ \alpha_{\psi(1)} = 3,\ \ \beta_{\psi(1)} = 3,\ \ \alpha_{\psi(2)} = 1,\ \ \beta_{\psi(2)} = 2 .$$

Further we note that $(R{-}i)$ and $(L{-}i)$ $(\,i = 1, 2\,)$ are not cyclic by Theorem 12 since $A \neq 0$ and $B \neq 0$.

(2) Suppose that $(\tilde{e}_1\tilde{R},\ \tilde{R}\tilde{e}_1)$ and $(\tilde{e}_2\tilde{R}, \tilde{R}\tilde{e}_2)$ are i-pairs. We consider

$$R \overset{put}{:=} \begin{pmatrix} Q_1 & Q_1 & \overline{Q}_1 & A & A \\ J(Q_1) & Q_1 & \overline{Q}_1 & A & A \\ J(Q_1) & J(Q_1) & Q_1 & A & A \\ B & B & B & Q_2 & Q_2 \\ B & B & B & J(Q_2) & Q_2 \end{pmatrix},$$

where we put $\overline{Q}_1 \overset{put}{:=} Q_1/S(Q_1)$ and, for each $j = 1, 2, \ldots, 5$, let e_j be the j-th matrix unit. Then R is a two-sided Harada ring as follows.

For instance, we put $e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{3,1},\ e_{3,2}$ as in (1). Then $\{ e_{1,1},\ e_{1,2},\ e_{2,1},\ e_{3,1},\ e_{3,2} \}$ is a left well-indexed set and $\alpha_i,\ \beta_i,\ (R{-}i),\ (L{-}i),\ f_{i,j}\ \ (i = 1, 2,\ j = 1, 2, 3)$ are the same as in (1). And

$$\delta_1 = 2,\ \delta_2 = 1,\ \ p_1(1) = 2,\ p_1(2) = 3,\ p_2(1) = 2\ \text{ and }\ q_1(1) = 1,\ q_1(2) = 2,\ q_2(1) = 3\,,$$

i.e.,

$$(e_{1,1}R,\ Rf_{1,2}),\ (e_{2,1}R,\ Rf_{1,3}),\ (e_{3,1}R,\ Rf_{2,2})$$

are i-pairs. So

$$\Psi(\,(L{-}1\,)\,) = [\,R{-}1\,] = (\,R{-}1\,)\ \text{ and }\ \Psi(\,(L{-}2\,)\,) = [\,R{-}2\,] = (\,R{-}2\,)\,.$$

Therefore

$$\psi(1) = 1,\ \psi(2) = 2\ \text{ and }\ \alpha_{\psi(1)} = 1,\ \beta_{\psi(1)} = 2,\ \alpha_{\psi(2)} = 3,\ \beta_{\psi(2)} = 3\,.$$

4. Left QF-well-indexed Set of QF Rings

Left QF-well-indexed sets have the following equivalent conditions.

Lemma 17. *Let Q be an indecomposable basic QF ring and let $\{ f'_{i,s} \}_{i=1,s=1}^{m'\ \delta'_i}$ be a complete set of orthogonal primitive idempotents of Q which satisfies* (QFWI 2). *The following are equivalent.*

(*a*) $\{ f'_{i,s} \}_{i=1,s=1}^{m'\ \delta'_i}$ *satisfies* (QFWI 1), *i.e.,* $\{ f'_{i,s} \}_{i=1,s=1}^{m'\ \delta'_i}$ *is a left QF-well-indexed set of Q.*

(*b*) (*i*) *If $\delta'_i \geq 2$, then ${}_QQf'_{i,s}/S({}_QQf'_{i,s}) \cong {}_QJ(Q)f'_{i,s+1}$ for any $s = 1, 2, \ldots, \delta'_i - 1$.*

(*ii*) *For any $i = 1, 2, \ldots, m'$ and $f \in \{ f'_{j,t} \}_{j=1,t=1}^{m'\ \delta'_j} - \{ f'_{i,s} \}_{s=1}^{\delta'_i}$ with $(fQ,\, Qf)$ an i-pair, both ${}_QQf/S({}_QQf) \ncong {}_QJ(Q)f'_{i,1}$ and ${}_QQf'_{i,\delta'_i}/S({}_QQf'_{i,\delta'_i}) \ncong {}_QJ(Q)f$ hold.*

(*a′*) $f'_{i,\delta'_i}Q,\ f'_{i,\delta'_i-1}Q,\ \ldots,\ f'_{i,1}Q$ *is a right w-co-H-sequence for any $i = 1, 2, \ldots, m'$.*

(*b′*) (*i*) *If $\delta'_i \geq 2$, then $f'_{i,s+1}Q_Q/S(f'_{i,s+1}Q_Q) \cong f'_{i,s}J(Q)_Q$ for any $s = 1, 2, \ldots, \delta'_i - 1$.*

(*ii*) *For any $i = 1, 2, \ldots, m'$ and $f \in \{ f'_{j,t} \}_{j=1,t=1}^{m'\ \delta'_j} - \{ f'_{i,s} \}_{s=1}^{\delta'_i}$ with $(fQ,\, Qf)$ an i-pair, both $fQ_Q/S(fQ_Q) \ncong f'_{i,\delta'_i}J(Q)_Q$ and $f'_{i,1}Q/S(f'_{i,1}Q_Q) \ncong fJ(Q)_Q$ hold.*

Further left QF-well-indexed sets have the following properties.

Lemma 18. *Let Q be an indecomposable basic QF ring with a left QF-well-indexed set $\{ f'_{i,s} \}_{i=1,s=1}^{m'\ \delta'_i}$. Then, since QF rings are two-sided Harada rings, bijection $\psi : \{ 1, 2, \ldots, m' \} \to \{ 1, 2, \ldots, m' \}$ given in* [2, §3] is defined. With respect to ψ, the following hold:

(1) For any $i = 1, 2, \ldots, m'$ and any $s = 1, 2, \ldots, \delta'_i$, $\big(f'_{\psi(i),s}Q,\ Qf'_{i,s}\big)$ is an i-pair. So $S\big(\, f'_{\psi(i),s}Qf'_{i,s}\,\big)$ is defined.

(2) In particular, if $\delta'_i \geq 2$, then $\psi(i) = i$.

(3) If $\delta'_i = 1$, then $\delta'_{\psi(i)} = 1$.

Let Q be an indecomposable basic QF ring with a left QF-well-indexed set $\{ f'_{i,s} \}_{i=1,s=1}^{m' \ \delta'_i}$. For each $i \in \{ 1, 2, \ldots, m' \}$, we put

$$r'_i(1) = 1, \quad x_{i,1} = 1$$

and we take positive integers

$$\delta_i \,, \qquad \gamma_i$$

to satisfy

$$\delta'_i \leq \delta_i \leq \gamma_i \,.$$

Morevoer, we take

$$r_i(u),\ p_i(u) \in \{ 1, 2, \ldots, \gamma_i \} \quad (u = 1, 2, \ldots, \delta_i)$$

to satisfy the following (1),(2),(3):

(1) The following (†-1) holds.

$$(\dagger\text{-}1) \quad \begin{array}{ll} (i) & 1 \leq p_i(1) < p_i(2) < \cdots < p_i(\delta_i) = \gamma_i \\ (ii) & 1 = r_i(1) < r_i(2) < \cdots < r_i(\delta_i) \leq \gamma_i \ \big(\text{So } \ r_i(x_{i,1}) = r'_i(1) = 1. \big) \end{array}$$

(2) If $\delta'_i = 1$ and $i = \psi(i)$, then the following (†-2) holds.

$$(\dagger\text{-}2) \quad r_i(u) \leq p_i(u-1) \text{ for all } u = 2, 3, \ldots, \gamma_i \,.$$

(3) If $\delta'_i \geq 2$ (we note that, then $i = \psi(i)$ from Lemma 18 (2)), then the following (†-3) holds, where we let

$$\begin{cases} r'_i(s) \in \{ 1, 2, \ldots, \gamma_i \} & (s = 2, 3, \ldots, \delta'_i) \\ p'_i(t) \in \{ 1, 2, \ldots, \gamma_i \} & (t = 1, 2, \ldots, \delta'_i - 1) \\ x_{i,s} \in \{ 2, 3, \ldots, \delta_i \} & (s = 1, 2, \ldots, \delta'_i) \\ y_{i,t} \in \{ 1, 2, \ldots, \delta_i - 1 \} & (t = 1, 2, \ldots, \delta'_i - 1) \,. \end{cases}$$

$$\begin{array}{lll} (\dagger\text{-}3)\ (i) & 1 = x_{i,1} \leq y_{i,1} < x_{i,2} \leq y_{i,2} < \cdots < x_{i,\delta'_i-1} \leq y_{i,\delta'_i-1} < x_{i,\delta'_i} & \\ (ii) & r_i(x_{i,s}) = r'_i(s) & (s = 2, 3, \ldots, \delta'_i) \\ (iii) & p_i(y_{i,t}) = p'_i(t) & (t = 1, 2, \ldots, \delta'_i - 1) \\ (iv) & p_i(x_{i,s} - 1) < r_i(x_{i,s}) \leq p_i(x_{i,s}) & (s = 2, 3, \ldots, \delta'_i) \\ (v) & r_i(y_{i,t}) \leq p_i(y_{i,t}) < r_i(y_{i,t} + 1) & (t = 1, 2, \ldots, \delta'_i - 1) \\ (vi) & r_i(u+1) \leq p_i(u) & \left(\begin{array}{l} x_{i,t} \leq u < y_{i,t} \,, \\ \text{where } \ t = 1, 2, \ldots, \delta'_i - 1 \end{array} \right) \\ (vii) & p_i(u) < r_i(u) & \left(\begin{array}{l} y_{i,t} < u < x_{i,t+1} \,, \\ \text{where } \ t = 1, 2, \ldots, \delta'_i - 1 \end{array} \right) \end{array}$$

Then the following holds.

Lemma 19. *Let Q be an indecomposable basic QF ring with a left QF-well-indexed set $\{ f'_{i,s} \}_{i=1,s=1}^{m' \ \delta'_i}$. Suppose that $\delta'_i \geq 2$. Then*

$$1 = r'_i(1) \leq p'_i(1) < r'_i(2) \leq p'_i(2) < \cdots < r'_i(\delta'_i - 1) \leq p'_i(\delta'_i - 1) < r'_i(\delta'_i) \leq \gamma_i \,.$$

5. Two sided Harada rings constructed from QF rings

For each $i = 1, 2, \dots, m'$ and $s = 1, 2, \dots, \gamma_i$, we put

$$\tau'_i(\, s\,) \overset{put}{:=} \max\{\; u \in \{\,1, 2, \dots, \delta_i\,\} \;\big|\; r_i(\,u\,) \leq s\,\}\,.$$

That is, $\tau'_i(\,s\,) \in \{\,1, 2, \dots, \delta_i\,\}$ such that

$$r_i\big(\,\tau'_i(s)\big) \leq s < r_i\big(\,\tau'_i(s) + 1\,\big)\,,$$

where we let $r_i\big(\,\delta_i + 1\,\big) = \gamma_i + 1\,.$

Now we construct two-sided Harada rings. Let Q be an indecomposable basic QF ring with a left QF-well-indexed set $\{\,f'_{i,s}\,\}_{i=1,s=1}^{m'\ \delta'_i}$ and we use the terminologies that now we define. For each $i,\, j = 1, 2, \dots, m'$, $k = 1, 2, \dots, \delta'_i$ and $l = 1, 2, \dots, \delta'_j$, we put

$$Q_{i,k;j,l} \overset{put}{:=} f'_{i,k} Q f'_{j,l}\,.$$

And we put

$$Q_{i,k} \overset{put}{:=} Q_{i,k;i,k}, \qquad J_{i,k} \overset{put}{:=} J(Q_{i,k}), \qquad S_{\psi(j),l;j,l} \overset{put}{:=} S(\,f'_{\psi(j),l} Q f'_{j,l}\,).$$

(We note that $S_{\psi(j),l;j,l}$ is defined by Lemma 18 (1).) Moreover, we put

$$m_{i,k} \overset{put}{:=} r'_i(\,k+1\,) - r'_i(\,k\,),$$

where we let $r'_i(\,\delta'_i + 1\,) = \gamma_i + 1$,

$$\mathbb{Q}_{i,k;j,l} \overset{put}{:=} \begin{cases} \begin{pmatrix} Q_{i,k} & \cdots & \cdots & Q_{i,k} \\ J_{i,k} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J_{i,k} & \cdots & J_{i,k} & Q_{i,k} \end{pmatrix} & \text{if } (i,k) = (j,l) \\ & \qquad : (m_{i,k},\, m_{j,l})\text{-matrix}, \\ \begin{pmatrix} Q_{i,k;j,l} & \cdots & Q_{i,k;j,l} \\ \vdots & & \vdots \\ Q_{i,k;j,l} & \cdots & Q_{i,k;j,l} \end{pmatrix} & \text{if } (i,k) \neq (j,l) \end{cases}$$

$$\mathbb{M}_{i,j} \overset{put}{:=} \begin{pmatrix} \mathbb{Q}_{i,1;j,1} & \mathbb{Q}_{i,1;j,2} & \cdots & \mathbb{Q}_{i,1;j,\delta'_j} \\ \mathbb{Q}_{i,2;j,1} & \mathbb{Q}_{i,2;j,2} & \cdots & \mathbb{Q}_{i,2;j,\delta'_j} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{Q}_{i,\delta'_i;j,1} & \mathbb{Q}_{i,\delta'_i;j,2} & \cdots & \mathbb{Q}_{i,\delta'_i;j,\delta'_j} \end{pmatrix} : \ (\,\gamma_i,\, \gamma_j)\text{-matrix},$$

(then we note that the (p,q)-component of $\mathbb{Q}_{i,k;j,l}$ is the $\big(\,r'_i(k) + p - 1,\, r'_j(l) + q - 1\,\big)$-component of $\mathbb{M}_{i,j}$) and

$$\tilde{R} \overset{put}{:=} \begin{pmatrix} \mathbb{M}_{1,1} & \mathbb{M}_{1,2} & \cdots & \mathbb{M}_{1,m'} \\ \mathbb{M}_{2,1} & \mathbb{M}_{2,2} & \cdots & \mathbb{M}_{2,m'} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{m',1} & \mathbb{M}_{m',2} & \cdots & \mathbb{M}_{m',m'} \end{pmatrix}.$$

Further, for each $p = 1, 2, \dots, m_{i,k}$ and $q = 1, 2, \dots, m_{j,l}$, we put

$$A_{i,k;j,l}^{\;p,q} \overset{put}{:=} \begin{cases} S_{i,k;j,l} & \text{if } \ i=\psi(j), \ \ k=l \ \text{ and } \\ & \quad p_j\big(\tau'_{\psi(j)}\big(r'_{\psi(j)}(k)+p-1\big)\big) < r'_j(l)+q-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbb{A}_{i,k;j,l} \overset{put}{:=} \begin{pmatrix} A_{i,k;j,l}^{\;1,1} & A_{i,k;j,l}^{\;1,2} & \cdots & A_{i,k;j,l}^{\;1,m_{j,l}} \\ A_{i,k;j,l}^{\;2,1} & A_{i,k;j,l}^{\;2,2} & \cdots & A_{i,k;j,l}^{\;2,m_{j,l}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i,k;j,l}^{\;m_{i,k},1} & A_{i,k;j,l}^{\;m_{i,k},2} & \cdots & A_{i,k;j,l}^{\;m_{i,k},m_{j,l}} \end{pmatrix} \ (\ :\ \text{subset of } \mathbb{Q}_{i,k;j,l}.)$$

For example, when $\delta'_i = \delta'_j = 1$ and $i=\psi(j)$, we put $S \overset{put}{:=} S_{i,1;j,1}$, and

$$\mathbb{A}_{i,1;j,1} = \begin{array}{r} \\ r_i(1) \\ \\ \\ \\ r_i(2) \\ \\ \\ \\ r_i(3) \\ \\ \\ \\ \vdots \\ r_i(\delta_i-1) \\ \\ \\ \\ r_i(\delta_i) \\ \\ \\ \\ \end{array} \begin{array}{l} \quad\quad\quad p_j(1) \quad\quad p_j(2) \quad\quad p_j(3) \quad \cdots \quad p_j(\delta_j-1) \quad p_j(\delta_j) \\ \left(\begin{array}{cccccccccc} 0 & \cdots\ 0 & S\ \cdots & & & & & & & \cdots\ S \\ 0 & \cdots\ 0 & S\ \cdots & & & & & & & \cdots\ S \\ \vdots & \vdots & \vdots & & & & & & & \vdots \\ 0 & \cdots\ 0 & S\ \cdots & & & & & & & \cdots\ S \\ 0 & \cdots\cdots & \cdots\ 0 & S\ \cdots & & & & & & \cdots\ S \\ 0 & \cdots\cdots & \cdots\ 0 & S\ \cdots & & & & & & \cdots\ S \\ \vdots & & \vdots & \vdots & & & & & & \vdots \\ 0 & \cdots\cdots & \cdots\ 0 & S\ \cdots & & & & & & \cdots\ S \\ 0 & \cdots\cdots & & \cdots\ 0 & S\cdots & & & & & \cdots\ S \\ 0 & \cdots\cdots & & \cdots\ 0 & S\cdots & & & & & \cdots\ S \\ \vdots & & & \vdots & \vdots & & & & & \vdots \\ 0 & \cdots\cdots & & \cdots\ 0 & S\cdots & & & & & \cdots\ S \\ \vdots & & & & & \ddots & & & & \vdots \\ 0 & \cdots\cdots & & & & & \cdots\ 0 & S & & \cdots\ S \\ 0 & \cdots\cdots & & & & & \cdots\ 0 & S & & \cdots\ S \\ \vdots & & & & & & \vdots & \vdots & & \vdots \\ 0 & \cdots\cdots & & & & & \cdots\ 0 & S & & \cdots\ S \\ 0 & \cdots\cdots & & & & & & & & \cdots\ 0 \\ 0 & \cdots\cdots & & & & & & & & \cdots\ 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots\cdots & & & & & & & & \cdots\ 0 \end{array}\right) \end{array} .$$

When $\delta'_i \geq 2$ and $i = \psi(i)$, we put $S' \overset{put}{:=} S_{i,k;i,k}$, and for $k = 1, 2, \ldots, \delta'_i - 1$

$$
\mathbb{A}_{i,k;i,k} \; = \;
\begin{array}{r}
 \\
r'_i(k) = r_i(x_{i,k}) \\
 \\
r_i(x_{i,k}+1) \\
\vdots \\
r_i(y_{i,k}) \\
 \\
 \\
r_i(y_{i,k}+1) \\
\vdots \\
r_i(x_{i,k+1}-1) \\
 \\
\end{array}
\left(
\begin{array}{cccccccccccc}
 & & a & & b \cdots c & & d \cdots e & & & & & \\
0 & \cdots \cdots & 0 & S' & \cdots & & & & & & \cdots & S' \\
0 & \cdots \cdots & 0 & S' & \cdots & & & & & & \cdots & S' \\
0 & \cdots & & & \cdots & 0 & S' & \cdots & & & \cdots & S' \\
0 & \cdots & & & \cdots & 0 & S' & \cdots & & & \cdots & S' \\
0 & \cdots & & & & \cdots & 0 & S' & \cdots & & \cdots & S' \\
\vdots & & & & & & \vdots & \vdots & \ddots & & & \vdots \\
0 & \cdots & & & & \cdots & 0 & S' & \cdots & & \cdots & S' \\
0 & \cdots & & & & & \cdots & 0 & S' & \cdots & \cdots & S' \\
0 & \cdots & & & & & \cdots & 0 & S' & \cdots & \cdots & S' \\
0 & \cdots & & & & & & \cdots & 0 & S' \; \cdots & \cdots & S' \\
0 & \cdots & & & & & & \cdots & 0 & S' \; \cdots & & S' \\
\end{array}
\right),
$$

where we put $a \overset{put}{:=} p_i(x_{i,k})$, $b \overset{put}{:=} p_j(x_{i,k}+1)$, $c \overset{put}{:=} p_j(y_{i,k})$, $d \overset{put}{:=} p_j(y_{i,k}+1)$ and $e \overset{put}{:=} p_j(x_{i,k+1}-1)$, and

$$
\mathbb{A}_{i,\delta'_i;i,\delta'_i} \; = \;
\begin{array}{r}
 \\
r'_i(\delta'_i) = r_i(x_{i,\delta'_i}) \\
 \\
r_i(x_{i,\delta'_i}+1) \\
 \\
r_i(x_{i,\delta'_i}+2) \\
\vdots \\
r_i(\delta_i - 1) \\
 \\
r_i(\delta_i) \\
 \\
\end{array}
\left(
\begin{array}{ccccccccccc}
 & & a' & & b' & & c' \cdots d' & & & & e' \\
0 & \cdots \cdots & 0 & S' \cdots & & & & & & \cdots & S' \\
0 & \cdots \cdots & 0 & S' \cdots & & & & & & \cdots & S' \\
0 & \cdots & & \cdots & 0 & S' \cdots & & & & \cdots & S' \\
0 & \cdots & & \cdots & 0 & S' \cdots & & & & \cdots & S' \\
0 & \cdots & & & & \cdots & 0 & S' \cdots & & \cdots & S' \\
0 & \cdots & & & & \cdots & 0 & S' \cdots & & \cdots & S' \\
0 & \cdots & & & & & & \cdots & 0 \; S' & \cdots & S' \\
0 & \cdots & & & & & & \cdots & 0 \; S' & \cdots & S' \\
0 & \cdots & & & & & & & & \cdots & 0 \\
\vdots & & & & & & & & \ddots & & \vdots \\
0 & \cdots & & & & & & & & \cdots & 0 \\
\end{array}
\right),
$$

where we put $a' \overset{put}{:=} p_i(x_{i,\delta'_i})$, $b' \overset{put}{:=} p_i(x_{i,\delta'_i}+1)$, $c' \overset{put}{:=} p_i(x_{i,\delta'_i+2})$, $d' \overset{put}{:=} p_i(\delta_i - 1)$ and $e' \overset{put}{:=} p_i(\delta_i) = \gamma_i$.

For each $i, j = 1, 2, \ldots, m'$, we put

$$
\mathrm{N}_{i,j} \overset{put}{:=} \begin{pmatrix} \mathbb{A}_{i,1;j,1} & \mathbb{A}_{i,1;j,2} & \cdots & \mathbb{A}_{i,1;j,\delta'_j} \\ \mathbb{A}_{i,2;j,1} & \mathbb{A}_{i,2;j,2} & \cdots & \mathbb{A}_{i,2;j,\delta'_j} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{A}_{i,\delta'_i;j,1} & \mathbb{A}_{i,\delta'_i;j,2} & \cdots & \mathbb{A}_{i,\delta'_i;j,\delta'_j} \end{pmatrix} \quad (: \text{ subset of } \mathbb{M}_{i,j} \,)
$$

and

$$\tilde{I} \overset{put}{:=} \begin{pmatrix} \mathrm{N}_{1,1} & \mathrm{N}_{1,2} & \cdots & \mathrm{N}_{1,m'} \\ \mathrm{N}_{2,1} & \mathrm{N}_{2,2} & \cdots & \mathrm{N}_{2,m'} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{N}_{m',1} & \mathrm{N}_{m',2} & \cdots & \mathrm{N}_{m',m'} \end{pmatrix} \quad (\text{: subset of } \tilde{R}\ ).$$

And $\tilde{R}$ is an artinian ring by usual addition and multiplication of matrix and $\tilde{I}$ is its ideal since $S_{\psi(j),l;j,l}$ is simple both as a left $Q_{\psi(j),l}$-module and as a right $Q_{j,l}$-module and, for any $p' \le p$, $p_j\big(\,\tau'_{\psi(j)}\big(\,r'_{\psi(j)}(k)+p'-1\,\big)\,\big) \le p_j\big(\,\tau'_{\psi(j)}\big(\,r'_{\psi(j)}(k)+p-1\,\big)\,\big)$ by (†-1)(i). Hence we consider a factor ring

$$R \overset{put}{:=} \tilde{R}/\tilde{I}\,.$$

From the definition of $\tilde{R}$, an element $\tilde{r}$ of $\tilde{R}$ is

$$\tilde{r} = \big(\,\tilde{a}_{i,k;j,l}^{\;p,q}\,\big)_{i,j=1,\,k=1,\,l=1,\,p=1,\,q=1}^{m'\quad\ \delta'_i\quad\ \delta'_j\quad m_{i,k}\quad m_{j,l}}\,,$$

where $\tilde{a}_{i,k;j,l}^{\;p,q}$ $\;(\,p = 1,2,\ldots,m_{i,k},\quad q = 1,2,\ldots,m_{j,l}\,)$ is a (p,q)-component of $\mathbb{Q}_{i,k;j,l}$ $(\,k=1,2,\ldots,\delta'_i,\quad l=1,2,\ldots,\delta'_j\,)$ which is a part of $\mathbb{M}_{i,j}$. Further we put

$$\begin{cases} s \overset{put}{:=} r'_i(\,k\,)+p-1 \\ t \overset{put}{:=} r'_j(\,l\,)+q-1 \end{cases} \quad \text{and} \quad \tilde{a}_{i,s;j,t} \overset{put}{:=} \tilde{a}_{i,k;j,l}^{\;p,q}\,.$$

Then

$$\tilde{r} = \big(\,\tilde{a}_{i,s;j,t}\,\big)_{i,j=1,\,s=1,\,t=1}^{m'\quad\ \gamma_i\quad\ \gamma_j}\,.$$

So an element r of R is

$$r = \big(\,a_{i,k;j,l}^{\;p,q}\,\big)_{i,j=1,\,k=1,\,l=1,\,p=1,\,q=1}^{m'\quad\ \delta'_i\quad\ \delta'_j\quad m_{i,k}\quad m_{j,l}} = \big(\,a_{i,s;j,t}\,\big)_{i,j=1,\,s=1,\,t=1}^{m'\quad\ \gamma_i\quad\ \gamma_j}\,,$$

where we put

$$a_{i,k;j,l}^{\;p,q} \;=\; a_{i,s;j,t} \overset{put}{:=} \begin{cases} \tilde{a}_{i,s;j,t} + S_{i,k;j,l} & \text{if } i=\psi(j),\ \ k=l,\ \ p_j\big(\,\tau'_i(\,s\,)\,\big) < t \\ \tilde{a}_{i,s;j,t} & \text{otherwise.} \end{cases}$$

Furthermore we put

$$A_{i,s;j,t} \overset{put}{:=} A_{i,k;j,l}^{\;p,q}\,.$$

On the other hand, for any $i,j = 1,2,\ldots,m'$, $s = 1,2,\ldots,\gamma_i$ and $t = 1,2,\ldots,\gamma_j$, we take

$$\begin{cases} k_s \,\in\, \{\,1,2,\ldots,\delta'_i\,\}, & p_s \,\in\, \{\,1,2,\ldots,m_{i,k_s}\,\} \\ l_t \,\in\, \{\,1,2,\ldots,\delta'_j\,\}, & q_t \,\in\, \{\,1,2,\ldots,m_{j,l_t}\,\} \end{cases}$$

to satisfy

$$\begin{cases} s = r'_i(\,k_s\,)+p_s-1 \\ t = r'_j(\,l_t)+q_t-1\,. \end{cases}$$

And, for each $i = 1,2,\ldots,m'$ and $s = 1,2,\ldots,\gamma_i$, we define an element

$$\tilde{f}_{i,s} = \big(\,\tilde{a}_{i',s';j',t'}\,\big)_{i',j'=1,\,s'=1,\,t'=1}^{m'\quad\ \gamma_i\quad\ \gamma_j}$$

of $\tilde{R}$ by

$$\tilde{a}_{i',s';j',t'} = \begin{cases} 1_{Q_{i,k_s}} & \text{if } i' = j' = i \text{ and } s' = t' = s \\ 0_{Q_{i',k_{s'};j',l_{t'}}} & \text{otherwise,} \end{cases}$$

and an element $f_{s,t}$ of R by

$$f_{s,t} \overset{put}{:=} \tilde{f}_{s,t} + \tilde{I}\,.$$

Then

$$A_{i,s;j,t} = \begin{cases} S_{i,k_s;j,l_t} & \text{if } i = \psi(j),\ \ k_s = l_t \ \text{ and } \ p_j\big(\tau'_i(\,s\,)\big) < t \\ 0 & \text{otherwise.} \end{cases}$$

Hence, from the definition of R, $f_{i,s}Rf_{j,t}$ is as follows:

(1) We assume that $i = j$.

(i) Suppose that $\delta'_i \geq 2$. (Then $i = \psi(i)$ by Lemma 18 (2).)

In the case $k_s = l_t$,

$$f_{i,s}Rf_{i,t} = \begin{cases} Q_{i,k_s} & \text{if } s \leq t \text{ and } p_i\big(\tau'_i(\,s\,)\big) \geq t \\ Q_{i,k_s}/S_{i,k_s} & \text{if } s \leq t \text{ and } p_i\big(\tau'_i(\,s\,)\big) < t \\ J_{i,k_s} & \text{if } s > t \text{ and } p_i\big(\tau'_i(\,s\,)\big) \geq t \\ J_{i,k_s}/S_{i,k_s} & \text{if } s > t \text{ and } p_i\big(\tau'_i(\,s\,)\big) < t. \end{cases}$$

In the case $k_s \neq l_t$, $f_{i,s}Rf_{i,t} = Q_{i,k_s;i,l_t}\,$.

(ii) Suppose that $\delta'_i = 1$. (Then $k_s = l_t = 1$ for any $s, t = 1, 2, \ldots, \gamma_i\,$.)

In the case $i = \psi(i)$, $f_{i,s}Rf_{i,t}$ coincides with one in the case (i) $k_s = l_t$.

In the case $i \neq \psi(i)$,

$$f_{i,s}Rf_{i,t} = \begin{cases} Q_{i,1} & \text{if } s \leq t \\ J_{i,1} & \text{if } s > t. \end{cases}$$

(2) Next we assume that $i \neq j$.

(i) Suppose that $\delta'_j \geq 2$. Then $f_{i,s}Rf_{j,t} = Q_{i,k_s;j,l_t}$.

(ii) Suppose that $\delta'_j = 1$.

In the case $i = \psi(j)$,

$$f_{i,s}Rf_{j,t} = \begin{cases} Q_{i,k_s;j,l_t} & \text{if } p_j\big(\tau'_i(\,s\,)\big) \geq t \\ Q_{i,k_s;j,l_t}/S_{i,k_s;j,l_t} & \text{if } p_j\big(\tau'_i(\,s\,)\big) < t. \end{cases}$$

In the case $i \neq \psi(j)$, (We note that, if $\delta'_i \geq 2$, then $i \neq \psi(j)$ by Lemma 18 (2).)

$$f_{i,s}Rf_{j,t} = Q_{i,k_s;j,l_t}\,.$$

Throughout this paper, we use these terminologies.

For each $i, j = 1, 2, \ldots, m'$, we consider the following sequences, where we let $p_j(0) = 0$ and $r_i(\delta_i + 1) = \gamma_i + 1$.

$$\begin{array}{ll} (L\text{-}j\text{-}u) & Rf_{j,p_j(u-1)+1},\ Rf_{j,p_j(u-1)+2},\ \ldots,\ Rf_{j,p_j(u)} \quad (u = 1, 2, \ldots, \delta_j) \\ (L\text{-}j) & Rf_{j,1},\ Rf_{j,2},\ \ldots,\ Rf_{j,\gamma_j} \\ (R\text{-}i\text{-}u) & f_{i,r_i(u+1)-1}R,\ f_{i,r_i(u+1)-2}R,\ \ldots,\ f_{i,r_i(u)}R \quad (u = 1, 2, \ldots, \delta_i) \\ (R\text{-}i) & f_{i,\gamma_i}R,\ f_{i,\gamma_i-1}R,\ \ldots,\ f_{i,1}R \end{array}$$

Theorem 20. *Then R is a two-sided Harada ring which satisfy the following:*

(1) $\left(f_{\psi(j),r_{\psi(j)}(u)}R,\ Rf_{j,p_j(u)}\right)$ *is an i-pair for any* $u = 1, 2, \ldots, \delta_j$.

(2) (i) *(L-j) is a left w-co-H-sequence for any* $j = 1, 2, \ldots, m'$.
 (ii) *(R-i) is a right w-co-H-sequence for any* $i = 1, 2, \ldots, m'$.

(3) (i) *(L-j-u) is a left co-H-sequence for any* $j = 1, 2, \ldots, m'$ *and* $u = 1, 2, \ldots, \delta_j$.
 (ii) *(R-i-u) is a right co-H-sequence for any* $i = 1, 2, \ldots, m'$ *and* $u = 1, 2, \ldots, \delta_i$.

Example 21. Let Q be an indecomposable basic QF ring such that

(i) its QF-well indexed set is $\{f'_{1,1}, f'_{1,2}, f'_{1,3}, f'_{2,1}, f'_{3,1}, f'_{4,1}\}$, and

(ii) $(f'_{1,1}Q, Qf'_{1,1})$, $(f'_{3,1}Q, Qf'_{2,1})$, $(f'_{2,1}Q, Qf'_{3,1})$, $(f'_{4,1}Q, Qf'_{4,1})$ are i-pairs.

The bijection $\psi : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ is defined by

$$\psi(1) = 1, \quad \psi(2) = 3, \quad \psi(3) = 2, \quad \psi(4) = 4$$

from (ii) above, and

$$\delta'_1 = 3, \qquad \delta'_2 = \delta'_3 = \delta'_4 = 1.$$

And for $i = 1, 2, 3, 4,$ we let, for instance, $\delta_i,\ \gamma_i\ p_i(u)$ and $r_i(u)\ (u = 1, 2, \ldots, \delta_i)$ as follows.

- $\delta_1 = 5, \quad \delta_2 = \delta_3 = \delta_4 = 2.$
- $\gamma_1 = 9, \quad \gamma_2 = \gamma_3 = 2, \quad \gamma_4 = 3.$
- $p_1(1) = 2, \quad p_1(2) = 3, \quad p_1(3) = 5, \quad p_1(4) = 6, \quad p_1(5) = 9$
- $r_1(1) = 1, \quad r_1(2) = 2, \quad r_1(3) = 5, \quad r_1(4) = 8, \quad r_1(5) = 9$
- $r_2(1) = 1, \quad r_2(2) = 2, \quad p_2(1) = 1, \quad p_2(2) = 2$
- $r_3(1) = 1, \quad r_3(2) = 2, \quad p_3(1) = 1, \quad p_3(2) = 2$
- $r_4(1) = 1, \quad r_4(2) = 2, \quad p_4(1) = 2, \quad p_4(2) = 3$

Then $\delta'_i \le \delta_i \le \gamma_i$, (†-1) and (†-2) hold. Further, for $s = 1, 2, 3\,(= \delta'_1)$ and $t = 1, 2\,(= \delta'_1 - 1)$, we let $r'_1(s)$, $p'_1(t)$, $x_{1,s}$, $y_{1,t}$ as follows:

- $r'_1(1) = 1, \quad r'_1(2) = 5, \quad r'_1(3) = 9$
- $p'_1(1) = 3, \quad p'_1(2) = 5$
- $x_{1,1} = 1, \quad x_{1,2} = 3, \quad x_{1,3} = 5$
- $y_{1,1} = 2, \quad y_{1,2} = 3$

Then (†-3) also holds. So, by Theorem 20, we can construct a two-sided Harada ring R with i-pairs

$$\begin{aligned}
&(f_{1,1}R,\, Rf_{1,2}),\ \ (f_{1,2}R,\, Rf_{1,3}),\ \ (f_{1,5}R,\, Rf_{1,5}),\ \ (f_{1,8}R,\, Rf_{1,6}),\ \ (f_{1,9}R,\, Rf_{1,9})\\
&(f_{3,1}R,\, Rf_{2,1}),\ \ (f_{3,2}R,\, Rf_{2,2}),\\
&(f_{2,1}R,\, Rf_{3,1}),\ \ (f_{2,2}R,\, Rf_{3,2}),\\
&(f_{4,1}R,\, Rf_{4,2}),\ \ (f_{4,2}R,\, Rf_{4,3})\,.
\end{aligned}$$

And, putting $Q_{i,k} \overset{put}{:=} Q_{i,k;i,k}$, $J_{i,k} \overset{put}{:=} J(Q_{i,k})$, $Q_{i,k;j,l} \overset{put}{:=} f'_{i,k}Qf'_{j,l}$ and $\overline{Q_{i,k;j,l}} \overset{put}{:=} Q_{i,k;j,l}/S(\,Q_{i,k;j,l}\,)$, R is isomorphic to

$$\begin{pmatrix}
Q_{11} & Q_{11} & \overline{Q_{11}} & \overline{Q_{11}} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;13} & Q_{11;21} & Q_{11;21} & Q_{11;31} & Q_{11;31} & Q_{11;41} & Q_{11;41} & Q_{11;41}\\
J_{11} & Q_{11} & Q_{11} & \overline{Q_{11}} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;13} & Q_{11;21} & Q_{11;21} & Q_{11;31} & Q_{11;31} & Q_{11;41} & Q_{11;41} & Q_{11;41}\\
J_{11} & J_{11} & Q_{11} & \overline{Q_{11}} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;13} & Q_{11;21} & Q_{11;21} & Q_{11;31} & Q_{11;31} & Q_{11;41} & Q_{11;41} & Q_{11;41}\\
J_{11} & J_{11} & J_{11} & \overline{Q_{11}} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;12} & Q_{11;13} & Q_{11;21} & Q_{11;21} & Q_{11;31} & Q_{11;31} & Q_{11;41} & Q_{11;41} & Q_{11;41}\\
Q_{12;11} & Q_{12;11} & Q_{12;11} & Q_{12;11} & Q_{12} & \overline{Q_{12}} & \overline{Q_{12}} & \overline{Q_{12}} & Q_{12;13} & Q_{12;21} & Q_{12;21} & Q_{12;31} & Q_{12;31} & Q_{12;41} & Q_{12;41} & Q_{12;41}\\
Q_{12;11} & Q_{12;11} & Q_{12;11} & Q_{12;11} & J_{12} & \overline{Q_{12}} & \overline{Q_{12}} & \overline{Q_{12}} & Q_{12;13} & Q_{12;21} & Q_{12;21} & Q_{12;31} & Q_{12;31} & Q_{12;41} & Q_{12;41} & Q_{12;41}\\
Q_{12;11} & Q_{12;11} & Q_{12;11} & Q_{12;11} & J_{12} & \overline{J_{12}} & \overline{Q_{12}} & \overline{Q_{12}} & Q_{12;13} & Q_{12;21} & Q_{12;21} & Q_{12;31} & Q_{12;31} & Q_{12;41} & Q_{12;41} & Q_{12;41}\\
Q_{12;11} & Q_{12;11} & Q_{12;11} & Q_{12;11} & J_{12} & J_{12} & \overline{J_{12}} & \overline{Q_{12}} & Q_{12;13} & Q_{12;21} & Q_{12;21} & Q_{12;31} & Q_{12;31} & Q_{12;41} & Q_{12;41} & Q_{12;41}\\
Q_{13;11} & Q_{13;11} & Q_{13;11} & Q_{13;11} & Q_{13;12} & Q_{13;12} & Q_{13;12} & Q_{13;12} & Q_{13} & Q_{13;21} & Q_{13;21} & Q_{13;31} & Q_{13;31} & Q_{13;41} & Q_{13;41} & Q_{13;41}\\
Q_{21;11} & Q_{21;11} & Q_{21;11} & Q_{21;11} & Q_{21;12} & Q_{21;12} & Q_{21;12} & Q_{21;12} & Q_{21;13} & Q_{21} & Q_{21} & Q_{21;31} & \overline{Q_{21;31}} & Q_{21;41} & Q_{21;41} & Q_{21;41}\\
Q_{21;11} & Q_{21;11} & Q_{21;11} & Q_{21;11} & Q_{21;12} & Q_{21;12} & Q_{21;12} & Q_{21;12} & Q_{21;13} & J_{21} & Q_{21} & Q_{21;31} & Q_{21;31} & Q_{21;41} & Q_{21;41} & Q_{21;41}\\
Q_{31;11} & Q_{31;11} & Q_{31;11} & Q_{31;11} & Q_{31;12} & Q_{31;12} & Q_{31;12} & Q_{31;12} & Q_{31;13} & Q_{31;21} & \overline{Q_{31;21}} & Q_{31} & Q_{31} & Q_{31;41} & Q_{31;41} & Q_{31;41}\\
Q_{31;11} & Q_{31;11} & Q_{31;11} & Q_{31;11} & Q_{31;12} & Q_{31;12} & Q_{31;12} & Q_{31;12} & Q_{31;13} & Q_{41;21} & Q_{31;21} & J_{31} & Q_{31} & Q_{31;41} & Q_{31;41} & Q_{31;41}\\
Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;13} & Q_{41;21} & Q_{41;21} & Q_{41;31} & Q_{41;31} & Q_{41} & Q_{41} & \overline{Q_{41}}\\
Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;13} & Q_{41;21} & Q_{41;21} & Q_{41;31} & Q_{41;31} & J_{41} & Q_{41} & Q_{41}\\
Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;11} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;12} & Q_{41;13} & Q_{41;21} & Q_{41;21} & Q_{41;31} & Q_{41;31} & J_{41} & J_{41} & Q_{41}
\end{pmatrix}.$$

6. QF ring $R(f)$ induced from two-sided Harada ring R and definitions of X_i, Y_i

Let $i,\, j \in \{\,1, 2, \ldots, m'\,\}$ and we assume that $i = \psi(j)$. Then we let

$$(\,f_{i,r_i(\,u\,)}R,\, Rf_{j,p_j(\,u\,)}\,)$$

be an i-pair for all $u = 1, 2, \ldots, \delta_j$.

Lemma 22. *Let $i,\, j = 1, 2, \ldots, m'$ with $i = \psi(j)$.*

(1) *For any $u = 1, 2, \ldots, \delta_j$ and $v_u = 1, 2, \ldots, n(q_j(u))$, the following hold.*

(I) $f_{i,\,r_i(u)+v_u-1} = e_{q_j(u),v_u}$. *So, in particular,* $f_{i,\,r_i(u)} = e_{q_j(u),1}$.

(II) *Suppose that* $r_i(1) = 1$. *And we put* $r_i(\delta_j + 1) = \gamma_i + 1$. *Then the following also hold.*

(*i*) $q_j(u) = \alpha_i + u - 1$.

(*ii*) $n(q_j(u)) = r_i(u+1) - r_i(u)$.

(*iii*) *The set of all right co-H-sequeces in* $(R\text{–}i)$ *is*

$$\begin{aligned} & \{ e_{q_j(u), n(q_j(u))}R,\ e_{q_j(u), n(q_j(u))-1}R,\ \dots,\ e_{q_j(u), 1}R \}_{u=1}^{\delta_j} \\ = & \{ f_{i, r_i(u+1)-1}R,\ f_{i, r_i(u+1)-2}R,\ \dots,\ f_{i, r_i(u)}R \}_{u=1}^{\delta_j} \\ = & \{ f_{i, r_i(u)+n(\alpha_i+u-1)-1}R,\ f_{i, r_i(u)+n(\alpha_i+u-1)-2}R,\ f_{i, r_i(u)+n(\alpha_i+u-1)-3}R, \\ & \dots,\ f_{i, r_i(u)}R \}_{u=1}^{\delta_j}. \end{aligned}$$

(*iv*) $1 = r_i(1) < r_i(2) < r_i(3) < \cdots < r_i(\delta_j)$.

(*v*) $r_i(u) = \begin{cases} 1 & \text{if } u = 1 \\ \sum_{s=1}^{u-1} n(\alpha_i + s - 1) + 1 & \text{if } u = 2, 3, \dots, \delta_j. \end{cases}$

(2) *Suppose that R is not a Nakayama ring. Then* $r_i(1) = 1$.

Let $i,\ j = 1, 2, \dots, m',\ s = 1, 2, \dots, \gamma_i$ and $t = 1, 2, \dots, \gamma_j$ and suppose that $r_i(1) = 1$. Then we put

$$\begin{aligned} \tau_j^l(t) &\overset{put}{:=} \min\{ u \in \{1, 2, \dots, \delta_j\} \mid t \le p_j(u) \} \\ \tau_i^r(s) &\overset{put}{:=} \max\{ u \in \{1, 2, \dots, \delta_{\psi^{-1}(i)} \mid r_i(u) \le s \} \end{aligned}$$

We note that

$$E({}_RRf_{j,t}) \cong {}_RRf_{j, p_j(\tau_j^l(t))} \quad \text{and} \quad E(f_{i,s}R_R) \cong f_{i, r_i(\tau_i^r(s))}R_R$$

and

$$p_j\left(\tau_j^l(t) - 1\right) < t \le p_j\left(\tau_j^l(t)\right) \quad \text{and} \quad r_i\left(\tau_i^r(s)\right) \le s < r_i\left(\tau_i^r(s) + 1\right),$$

where we let $p_j(0) = 0$ and $r_i\left(\delta_{\psi^{-1}(i)} + 1\right) = \gamma_i + 1$.

From here throughout this section, we suppose that $r_i(1) = 1$ holds for any $i = 1, 2, \dots, m'$. (For instance, when R is not a Nakayama ring by Lemma 22 (2).)

For each $i = 1, 2, \dots, m'$, we put

$$X_i \overset{put}{:=} \begin{cases} \{1\} \cup \left\{ u \in \{2, 3, \dots, \delta_i\} \;\middle|\; p_i(u-1) < r_i(u) \le p_i(u) \right\} & \text{if } \psi(i) = i \\ \{1\} & \text{if } \psi(i) \ne i. \end{cases}$$

Further we put

$$f_i \overset{put}{:=} \begin{cases} \sum_{u \in X_i} f_{i,r_i(u)} & \text{if } \psi(i) = i \\ f_{i,1} & \text{if } \psi(i) \neq i \,, \end{cases}$$

$$f \overset{put}{:=} \textstyle\sum_{i=1}^{m'} f_i \,.$$

Moreover, in the case $\psi(i) = i$, we put

$$\{ r_i(u) \}_{u \in X_i} = \{ r_i'(1), r_i'(2), \ldots, r_i'(\delta_i') \},$$

where we let $1 = r_i'(1) < r_i'(2) < \cdots < r_i'(\delta_i')$. (So $f_i = \sum_{k=1}^{\delta_i'} f_{i,r_i'(k)}$.)

And, in the case $\psi(i) \neq i$, we put

$$r_i'(1) \overset{put}{:=} 1, \quad \delta_i' \overset{put}{:=} 1 \,.$$

Theorem 23. *Then, for* $R(f)$ $(= fRf)$, *the following hold.*

(1) $\{ f_{i, r_i'(k)} \}_{i=1,k=1}^{m' \ \ \delta_i'}$ *is a complete set of orthogonal primitive idempotents of* $R(f)$ *such that, for each* $i = 1, 2, \ldots, m'$, *the following* $(i), (ii)$ *hold.*

- (i) $\big(f_{\psi(i),1} R(f),\, R(f) f_{i,1} \big)$ *is an i-pair.*
- (ii) *Suppose that* $\delta_i' \geq 2$. *Then* $\psi(i) = i$ *and* $\big(f_{i, r_i'(k)} R(f),\, R(f) f_{i, r_i'(k)} \big)$ *is an i-pair for any* $k = 1, 2, \ldots, \delta_i'$.

(2) $R(f)$ *is an indecomposable basic QF ring.*

For each $i \in \{ 1, 2, \ldots, m' \}$, the sequences

$R(f) f_{i,r_i'(1)},\ R(f) f_{i,r_i'(2)},\ \ldots,\ R(f) f_{i,r_i'(\delta_i')}$ and $f_{i,r_i'(\delta_i')} R(f),\ f_{i,r_i'(\delta_i'-1)} R(f),\ \ldots,\ f_{i,r_i'(1)} R(f)$

of left and right $R(f)$-modules are denoted by

$$(L\text{--}i)_{R(f)} \quad \text{and} \quad (R\text{--}i)_{R(f)} \,,$$

respectively.

Theorem 24. *For any* $i \in \{ 1, 2, \ldots, m' \}$, *the following hold.*

(1) (i) $(L\text{--}i)_{R(f)}$ *is a left w-co-H-sequence.*
(ii) $(L\text{--}i)$ *is cyclic if and only if* $(L\text{--}i)_{R(f)}$ *is so.*

(2) (i) $(R\text{--}i)_{R(f)}$ *is a right w-co-H-sequence.*
(ii) $(R\text{--}i)$ *is cyclic if and only if* $(R\text{--}i)_{R(f)}$ *is so.*

We put $f_{i,k}' \overset{put}{:=} f_{i,r_i'(k)}$ for any $i = 1, 2, \ldots, m'$ and $k = 1, 2, \ldots, \delta_i'$. Then $R(f)$ is an indecomposable basic QF ring with a complet set $\{ f_{i,k}' \}_{i=1,k=1}^{m' \ \delta_i'}$ of orthogonal primitive idempotents by Theorem 23 (2). Next we further show the following.

Corollary 25. $\{ f_{i,k}' \}_{i=1,k=1}^{m' \ \ \delta_i'}$ *is a left QF-well-indexed set of* $R(f)$.

For each $i = 1, 2, \ldots, m'$, we put

$$Y_i \overset{put}{:=} \begin{cases} \{\, u \in \{1, 2, \ldots, \delta_i - 1\} \;\big|\; r_i(\,u\,) \,\le\, p_i(\,u\,) \,<\, r_i(\,u+1\,) \,\} & (\text{if } \psi(i) = i\,) \\ \phi & (\text{if } \psi(i) \neq i\,) \end{cases}$$

And, in the case $\psi(i) = i$, we let

$$\{\, p_i(\,u\,)\,\}_{u \in Y_i} \;=\; \{\, p'_i(1),\, p'_i(2),\, \ldots,\, p'_i(\,\delta''_i\,)\,\},$$

with $p'_i(1) \,<\, p'_i(2) \,<\, \cdots \,<\, p'_i(\,\delta''_i\,)$.

We note that, from the definition of $p_i(1)$, $p_i(1), \ldots, p_i(\delta_i)$

$$p_i(1) < p_i(2) < \cdots < p_i(\delta_i)$$

holds. Further, if $r_i(1) = 1$, then

$$1 = r_i(1) < r_i(2) < \cdots < r_i(\delta_i)$$

also holds by [2, Theorem 3.3 (1)].

We let $i = 1, 2, \ldots, m'$. In the case $\psi(i) \neq i$ we put

$$x_{i,1} \overset{put}{:=} 1$$

And in the case $\psi(i) = i$ we put

- $X_i \overset{put}{:=} \{\, x_{i,1},\, x_{i,2},\, \ldots,\, x_{i,\delta'_i}\,\}$, where $x_{i,1} < x_{i,2} < \cdots < x_{i,\delta'_i}$. (So $x_{i,1} = 1$.)
- $Y_i \overset{put}{:=} \{\, y_{i,1},\, y_{i,2},\, \ldots,\, y_{i,\delta''_i}\,\}$, where $y_{i,1} < y_{i,2} < \cdots < y_{i,\delta''_i}$.

Then it is clear that the following hold from the definitions of X_i and Y_i.

(∗1) $p_i(\, x_{i,s} - 1\,) < r_i(\, x_{i,s}\,) \le p_i(\, x_{i,s}\,)$ for any $s = 2, 3, \ldots, \delta'_i$.

(∗2) $r_i(\, y_{i,t}\,) \le p_i(\, y_{i,t}\,) < r_i(\, y_{i,t} + 1\,)$ for any $t = 1, 2, \ldots, \delta''_i$, where we let $r_i(\,\delta_i + 1\,) \,=\, \gamma_i$.

Theorem 26. *We let $i \in \{\, 1, 2, \ldots, m'\,\}$ with $\psi(i) = i$. Then the following hold.*

(1) *Either $\delta''_i \,=\, \delta'_i - 1$ or $\delta''_i \,=\, \delta'_i$ holds*

(2) (*i*) *Suppose that $\delta''_i \,=\, \delta'_i - 1$, then*

$$1 \,=\, x_{i,1} \,\le\, y_{i,1} \,<\, x_{i,2} \,\le\, y_{i,2} \,<\, \cdots \,<\, x_{i,\delta'_i - 1} \,\le\, y_{i,\delta'_i - 1} \,<\, x_{i,\delta'_i}.$$

(*ii*) *Suppose that $\delta''_i \,=\, \delta'_i$, then*

$$1 \,=\, x_{i,1} \,\le\, y_{i,1} \,<\, x_{i,2} \,\le\, y_{i,2} \,<\, \cdots \,<\, x_{i,\delta'_i - 1} \,\le\, y_{i,\delta'_i - 1} \,<\, x_{i,\delta'_i} \,\le\, y_{i,\delta'_i}.$$

Theorem 27. *Suppose that $(\,L{-}i\,)$ is not cyclic. Then $\delta''_i \,=\, \delta'_i - 1$ holds.*

7. Matrix representation

Throughout this section, we assume that R is not Nakayama ring. Then we note that $\delta''_i = \delta'_i - 1$ by Theorem 12 and Proposition 27.

Lemma 28. *We let $i = 1, 2, \dots, m'$.*

(1) *The following (†-1) holds.*

(†-1) (*i*) $1 \leq p_i(1) < p_i(2) < \cdots < p_i(\,\delta_i\,) = \gamma_i$
(*ii*) $1 = r_i(1) < r_i(2) < \cdots < r_i(\,\delta_i\,) \leq \gamma_i$ $\big($ *So* $r_i(\,x_{i,1}\,) = r'_i(\,1\,) = 1.\big)$

(2) *If $\delta'_i = 1$ and $i = \psi(i)$, then the following condition (†-2) holds.*

(†-2) $r_i(\,u\,) \leq p_i(\,u - 1\,)$ *for all* $u = 2, 3, \dots, \gamma_i$.

(3) *If $\delta'_i \geq 2$ (we note that, then $i = \psi(i)$ from Theorem 23 (1)(ii)), the following (†-3) and (†) hold.*

(†-3) (*i*) $1 = x_{i,1} \leq y_{i,1} < x_{i,2} \leq y_{i,2} < \cdots < x_{i,\delta'_i-1} \leq y_{i,\delta'_i-1} < x_{i,\delta'_i}$
(*ii*) $r_i(\,x_{i,s}\,) = r'_i(\,s\,)$ $(\,s = 2, 3, \dots, \delta'_i\,)$
(*iii*) $p_i(\,y_{i,t}\,) = p'_i(\,t\,)$ $(\,t = 1, 2, \dots, \delta'_i - 1\,)$
(*iv*) $p_i(\,x_{i,s} - 1\,) < r_i(\,x_{i,s}\,) \leq p_i(\,x_{i,s}\,)$ $(\,s = 2, 3, \dots, \delta'_i\,)$
(*v*) $r_i(\,y_{i,t}\,) \leq p_i(\,y_{i,t}\,) < r_i(\,y_{i,t} + 1\,)$ $(\,t = 1, 2, \dots, \delta'_i - 1\,)$
(*vi*) $r_i(\,u + 1\,) \leq p_i(\,u\,)$ $\left(\begin{array}{l} x_{i,t} \leq u < y_{i,t}\,, \\ \textit{where } t = 1, 2, \dots, \delta'_i - 1 \end{array}\right)$
(*vii*) $p_i(\,u\,) < r_i(\,u\,)$ $\left(\begin{array}{l} y_{i,t} < u < x_{i,t+1}\,, \\ \textit{where } t = 1, 2, \dots, \delta'_i - 1 \end{array}\right)$

(†) $1 = r'_i(1) \leq p'_i(1) < r'_i(2) \leq p'_i(2) < r'_i(3) \leq p'_i(3) < \cdots < r'_i(\,\delta'_i - 1) \leq p'_i(\,\delta'_i - 1) < r'_i(\,\delta'_i)$

For each $i = 1, 2, \dots, m'$ and $k = 1, 2, \dots, \delta'_i$, we put

$$m_{i,k} \overset{put}{:=} r'_i(\,k + 1\,) - r'_i(\,k\,)\,,$$

where we let $r'_i(\,\delta'_i + 1\,) = \gamma_i + 1$.

For an indecomposable basic two-sided Harada ring R, by Theorem 23 (2) and Corollary 25, we see that $R(f) = fRf$ is an indecomposable QF ring with a left QF-well-indexed set $\{\,f'_{i,k}\,\}_{i=1,k=1}^{m'\ \ \delta'_i}$, where we put $f'_{i,k} \overset{put}{:=} f_{i,r'_i(k)}$. Further we obtain a bijection

$$\psi : \{\,1, 2, \dots, m'\,\} \to \{\,1, 2, \dots,\, m'\,\}$$

and, for each $i = 1, 2, \dots, m'$,

$p_i(u),\ r_i(u)$ $(\,u = 1, 2, \dots, \delta_i\,)$
$r'_i(s),\ x_{i,s}$ $(\,s = 1, 2, \dots, \delta'_i\,)$
$p'_i(t),\ y_{i,t}$ $(\,t = 1, 2, \dots, \delta'_i - 1\,)$

to satisfy the conditions just after Lemma 18 by Lemma 28. So, by §5, we have a two-sided Harada ring as follows:

For each $i,\ j=1,2,\ldots,m'$, $k=1,2,\ldots,\delta'_i$ and $l=1,2,\ldots,\delta'_j$, we put

$$Q_{i,k;j,l} \overset{put}{:=} f'_{i,k}Q(f)f'_{j,l}\ (=f'_{i,k}Rf'_{j,l}\,).$$

And we put

$$Q_{i,k} \overset{put}{:=} Q_{i,k;i,k}, \qquad J_{i,k} \overset{put}{:=} J(Q_{i,k}), \qquad S_{\psi(j),l;j,l} \overset{put}{:=} S(\,f'_{\psi(j),l}Qf'_{j,l}\,).$$

$$\mathbb{Q}_{i,k;j,l} \overset{put}{:=} \begin{cases} \begin{pmatrix} Q_{i,k} & \cdots & \cdots & Q_{i,k} \\ J_{i,k} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J_{i,k} & \cdots & J_{i,k} & Q_{i,k} \end{pmatrix} & \text{if } \ (i,k)=(j,l) \\ \begin{pmatrix} Q_{i,k;j,l} & \cdots & Q_{i,k;j,l} \\ \vdots & & \vdots \\ Q_{i,k;j,l} & \cdots & Q_{i,k;j,l} \end{pmatrix} & \text{if } \ (i,k)\neq(j,l) \end{cases} : (m_{i,k},\ m_{j,l})\text{-matrix},$$

$$\mathbb{M}_{i,j} \overset{put}{:=} \begin{pmatrix} \mathbb{Q}_{i,1;j,1} & \mathbb{Q}_{i,1;j,2} & \cdots & \mathbb{Q}_{i,1;j,\delta'_j} \\ \mathbb{Q}_{i,2;j,1} & \mathbb{Q}_{i,2;j,2} & \cdots & \mathbb{Q}_{i,2;j,\delta'_j} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{Q}_{i,\delta'_i;j,1} & \mathbb{Q}_{i,\delta'_i;j,2} & \cdots & \mathbb{Q}_{i,\delta'_i;j,\delta'_j} \end{pmatrix} : \ (\gamma_i,\ \gamma_j)\text{-matrix},$$

(then we note that the (p,q)-component of $\mathbb{Q}_{i,k;j,l}$ is the $\big(r'_i(k)+p-1,\ r'_j(l)+q-1\big)$-component of $\mathbb{M}_{i,j}$) and

$$\tilde{R} \overset{put}{:=} \begin{pmatrix} \mathbb{M}_{1,1} & \mathbb{M}_{1,2} & \cdots & \mathbb{M}_{1,m'} \\ \mathbb{M}_{2,1} & \mathbb{M}_{2,2} & \cdots & \mathbb{M}_{2,m'} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{M}_{m',1} & \mathbb{M}_{m',2} & \cdots & \mathbb{M}_{m',m'} \end{pmatrix}.$$

Further, for each $p=1,2,\ldots,m_{i,k}$ and $q=1,2,\ldots,m_{j,l}$, we put

$$A_{i,k;j,l}^{\ p,q} \overset{put}{:=} \begin{cases} S_{i,k;j,l} & \text{if } \ i=\psi(j),\ \ k=l \ \text{ and } \\ & \quad r'_j(l) \ \le\ p_j\big(\tau^r_{\psi(j)}\big(r'_{\psi(j)}(k)+p-1\big)\big) \ <\ r'_j(l)+q-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbb{A}_{i,k;j,l} \overset{put}{:=} \begin{pmatrix} A_{i,k;j,l}^{\ 1,1} & A_{i,k;j,l}^{\ 1,2} & \cdots & A_{i,k;j,l}^{\ 1,m_{j,l}} \\ A_{i,k;j,l}^{\ 2,1} & A_{i,k;j,l}^{\ 2,2} & \cdots & A_{i,k;j,l}^{\ 2,m_{j,l}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i,k;j,l}^{\ m_{i,k},1} & A_{i,k;j,l}^{\ m_{i,k},2} & \cdots & A_{i,k;j,l}^{\ m_{i,k},m_{j,l}} \end{pmatrix} \ (\,:\ \text{subset of } \mathbb{Q}_{i,k;j,l}).$$

For each $i,j=1,2,\ldots,m'$, we put

$$\mathrm{N}_{i,j} \overset{put}{:=} \begin{pmatrix} \mathbb{A}_{i,1;j,1} & \mathbb{A}_{i,1;j,2} & \cdots & \mathbb{A}_{i,1;j,\delta'_j} \\ \mathbb{A}_{i,2;j,1} & \mathbb{A}_{i,2;j,2} & \cdots & \mathbb{A}_{i,2;j,\delta'_j} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{A}_{i,\delta'_i;j,1} & \mathbb{A}_{i,\delta'_i;j,2} & \cdots & \mathbb{A}_{i,\delta'_i;j,\delta'_j} \end{pmatrix} \quad (:\ \text{subset of } \mathbb{M}_{i,j}\ )$$

and

$$\tilde{I} \overset{put}{:=} \begin{pmatrix} \mathrm{N}_{1,1} & \mathrm{N}_{1,2} & \cdots & \mathrm{N}_{1,m'} \\ \mathrm{N}_{2,1} & \mathrm{N}_{2,2} & \cdots & \mathrm{N}_{2,m'} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{N}_{m',1} & \mathrm{N}_{m',2} & \cdots & \mathrm{N}_{m',m'} \end{pmatrix} \quad (:\ \text{subset of } \tilde{R}\ ).$$

Then $\tilde{I}$ is an ideal of $\tilde{R}$ and the factor ring

$$R' \overset{put}{:=} \tilde{R}/\tilde{I}$$

is a two-sided Harada ring by [3, Theorem 3.1].

Theorem 29. *Then*

$$R \cong R' \quad \textit{as rings.}$$

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