SOME CLASSES OF SUBCATEGORIES OF MODULE CATEGORIES: CLASSIFICATIONS AND THE RELATION BETWEEN THEM

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Abstract. We introduce several classes of subcategories of module categories of artin algebras, and show that these classes of subcategories can be controlled by the poset of torsion classes in some sense. This enables us to compute and check various properties of the poset of such classes of subcategories by using computer.

1. Introduction

In the representation theory of algebras, the study of subcategories of module categories have been one of the main topics. Among them, *torsion classes* (subcategories closed under quotients and extensions) have been attracted an attention. There is another class of subcategories of module categories, *wide subcategories* (subcategories closed under kernels, cokernels, and extensions). Wide subcategories are related to ring epimorphisms of an algebra, and the relation between torsion classes and wide subcategories have been studied by several authors, e.g. [2], [6].

The motivating question of this article is as follows:

Question. *Are there any classes of subcategories of module categories which are related to torsion classes?*

In this article, we give such classes of subcategories. More precisely, we will see the following reconstruction theorem.

Theorem 1. *Suppose that the poset* $\mathsf{tors}\,\Lambda$ *of torsion classes is given as an abstract poset. Then, we can recover the posets of the following classes of subcategories of* $\mathsf{mod}\,\Lambda$*.*

1. *Wide subcategories.*
2. *ICE-closed subcategories.*
3. *IKE-closed subcategories.*
4. *Torsion hearts.*

We will see later the definitions of these subcategories.

1.1. Notation and conventions.

Notation and conventions. Throughout this article, we fix a filed k and a finite-dimensional k-algebra Λ. We denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated right Λ-modules. A *subcategory* is always assumed to be full and closed under isomorphisms.

The detailed version of this paper will be submitted for publication elsewhere.

2. Torsion(-free) classes and Serre and wide subcategories

In this section, we recall some classes of subcategories of $\mathsf{mod}\,\Lambda$ which have been previously investigated in the literature.

First of all, we recall the definition of torsion classes, which play a central role throughout this research.

Definition 2. Let Λ be a finite-dimensional k-algebra.

(1) A subcategory $\mathcal{T}$ of $\mathsf{mod}\,\Lambda$ is called a *torsion class* if it satisfies the following two conditions.
- $\mathcal{T}$ is *closed under extensions*, that is, for every short exact sequence
$$0 \to L \to M \to N \to 0$$
in $\mathsf{mod}\,\Lambda$, if $L, N \in \mathcal{T}$, then $M \in \mathcal{T}$.
- $\mathcal{T}$ is *closed under quotients*, that is, for every $T \in \mathcal{T}$, every quotient of T belongs to $\mathcal{T}$.

(2) Dually, a subcategory $\mathcal{F}$ is called a *torsion-free class* if it is closed under extensions and submodules.

We denote by $\mathsf{tors}\,\Lambda$ and $\mathsf{torf}\,\Lambda$ the posets of torsion classes and torsion-free classes in $\mathsf{mod}\,\Lambda$ respectively, ordered by inclusion.

This notion was first defined by Dickson [4] to generalize the notion of torsion abelian groups to general abelian categories. Recently, due to the important development in τ-tilting theory due to Adachi-Iyama-Reiten [1], the structure of the poset $\mathsf{tors}\,\Lambda$ has been studied very well. The aim of this research is to describe other classes of subcategories using this well-studied poset $\mathsf{tors}\,\Lambda$.

We remark that $\mathsf{tors}\,\Lambda$ and $\mathsf{torf}\,\Lambda$ are anti-isomorphic to each other by the maps $(-)^{\perp}\colon \mathsf{tors}\,\Lambda \to \mathsf{torf}\,\Lambda$ and ${}^{\perp}(-)\colon \mathsf{torf}\,\Lambda \to \mathsf{tors}\,\Lambda$, where

$$\mathcal{T}^{\perp} := \{M \in \mathsf{mod}\,\Lambda \mid \operatorname{Hom}_{\Lambda}(T, M) = 0 \text{ for every } T \text{ in } \mathcal{T}\}, \text{and}$$
$${}^{\perp}\mathcal{F} := \{M \in \mathsf{mod}\,\Lambda \mid \operatorname{Hom}_{\Lambda}(M, F) = 0 \text{ for every } F \text{ in } \mathcal{F}\}.$$

Next, we introduce two classes of abelian subcategories of $\mathsf{mod}\,\Lambda$.

Definition 3. Let Λ be a finite-dimensional k-algebra.

(1) A subcategory $\mathcal{S}$ of $\mathsf{mod}\,\Lambda$ is called a *Serre subcategory* if it is closed under extensions, quotients, and submodules.

(2) A subcategory $\mathcal{W}$ of $\mathsf{mod}\,\Lambda$ is called a *wide subcategory* if it satisfies the following conditions:
- $\mathcal{W}$ is closed under extensions.
- $\mathcal{W}$ is *closed under kernels*, that is, for every map $f\colon W_1 \to W_2$ with $W_1, W_2 \in \mathcal{W}$, we have $\operatorname{Ker} f \in \mathcal{W}$.
- $\mathcal{W}$ is *closed under cokernels*, that is, for every map $f\colon W_1 \to W_2$ with $W_1, W_2 \in \mathcal{W}$, we have $\operatorname{Coker} f \in \mathcal{W}$.

We denote by $\mathsf{Serre}\,\Lambda$ and $\mathsf{wide}\,\Lambda$ the poset of Serre subcategories and wide subcategories of $\mathsf{mod}\,\Lambda$ respectively.

It is clear from definitions that $\mathsf{Serre}\,\Lambda \subseteq \mathsf{wide}\,\Lambda$ and $\mathsf{tors}\,\Lambda \cap \mathsf{torf}\,\Lambda = \mathsf{Serre}\,\Lambda$ holds. As for the relation between wide subcategories and torsion classes, we mention the following surprising result due to Marks-Šťovíček [6]:

Theorem 4. *Suppose that* $\mathsf{tors}\,\Lambda$ *is a finite set. Then there exists a bijection between* $\mathsf{wide}\,\Lambda$ *and* $\mathsf{tors}\,\Lambda$*. For a wide subcategory* $\mathcal{W}$*, the corresponding torsion class is the smallest torsion class containing* $\mathcal{W}$*.*

Remark 5. The above bijection is *not a poset isomorphism*, thus $\mathsf{tors}\,\Lambda$ and $\mathsf{wide}\,\Lambda$ are not isomorphic in general. One of the motivation of this research is to investigate the poset structure of $\mathsf{wide}\,\Lambda$ using $\mathsf{tors}\,\Lambda$.

3. ICE-closed subcategories and torsion hearts

In this section, we give definitions and some properties of subcategories which the author introduced. Here a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$ is *closed under images* if for every map $f : C_1 \to C_2$ with $C_1, C_2 \in \mathcal{C}$, we have $\operatorname{Im} f \in \mathcal{C}$.

Definition 6. Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$.

(1) $\mathcal{C}$ is called an *ICE-closed subcategory* if it is closed under images, cokernels, and extensions.
(2) $\mathcal{C}$ is called an *IKE-closed subcategory* if it is closed under images, kernels, and extensions.

We denote by $\mathsf{ICE}\,\Lambda$ and $\mathsf{IKE}\,\Lambda$ the poset of ICE-closed and IKE-closed subcategories respectively.

It is immediate from definitions that every torsion class and wide subcategory is ICE-closed, thus $\mathsf{tors}\,\Lambda \subseteq \mathsf{ICE}\,\Lambda$ and $\mathsf{wide}\,\Lambda \subseteq \mathsf{ICE}\,\Lambda$ holds. Also we have $\mathsf{wide}\,\Lambda = \mathsf{ICE}\,\Lambda \cap \mathsf{IKE}\,\Lambda$.

The following figure describes the relation between several classes of subcategories appearing above.

$$\begin{array}{ccccc}
\mathsf{ICE}\,\Lambda & & & & \mathsf{IKE}\,\Lambda \\
& \supseteq & & \subseteq & \\
\cup & & \mathsf{wide}\,\Lambda & & \cup \\
& & & & \\
\mathsf{tors}\,\Lambda & & \cup & & \mathsf{torf}\,\Lambda \\
& \supseteq & & \subseteq & \\
& & \mathsf{Serre}\,\Lambda & &
\end{array}$$

Typical examples of ICE-closed subcategories are wide subcategories and torsion classes, and we can also easily check that every torsion class in a wide subcategory (regarded as an abelian category) is ICE-closed. The first result of this article is that the converse holds, hence we can use τ-tilting theory to classify ICE-closed subcategories.

Theorem 7 ([5])**.** *A subcategory* $\mathcal{C}$ *of* $\mathsf{mod}\,\Lambda$ *is ICE-closed if and only if there is some wide subcategory* $\mathcal{W}$ *of* $\mathsf{mod}\,\Lambda$ *containing* $\mathcal{C}$ *such that* $\mathcal{C}$ *is a torsion class in* $\mathcal{W}$*. Moreover, if* $\mathsf{tors}\,\Lambda$ *is a finite set, then there is a bijection between the following sets:*

(1) $\mathsf{ICE}\,\Lambda$.
(2) *The set of wide τ-tilting modules, that is, τ-tilting objects in some wide subcategory.*

Remark 8. It is an interesting problem to characterize wide τ-tilting modules homologically inside $\mathsf{mod}\,\Lambda$. For example, if Λ is hereditary, then wide τ-tilting modules are precisely rigid modules.

Finally, we introduce the following construction, which plays a central role in the latter part of this article.

Definition 9. Let $\mathcal{U}$ and $\mathcal{T}$ be two torsion classes in $\mathsf{mod}\,\Lambda$ satisfying $\mathcal{U} \subseteq \mathcal{T}$. Then the *heart of the interval* $[\mathcal{U}, \mathcal{T}]$ is the subcategory $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ defined as follows:

$$\mathcal{H}_{[\mathcal{U},\mathcal{T}]} = \mathcal{T} \cap \mathcal{U}^{\perp}$$

We call a subcategory of $\mathsf{mod}\,\Lambda$ of the form $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ a *torsion heart.* We denote by $\mathsf{tors\text{-}heart}\,\Lambda$ the poset of torsion hearts in $\mathsf{mod}\,\Lambda$ ordered by inclusion.

The subcategory $\mathcal{H}_{[\mathcal{U},\mathcal{T}]}$ represents a "difference $\mathcal{T} - \mathcal{U}$" in some sense. For example, we have the following equalities.

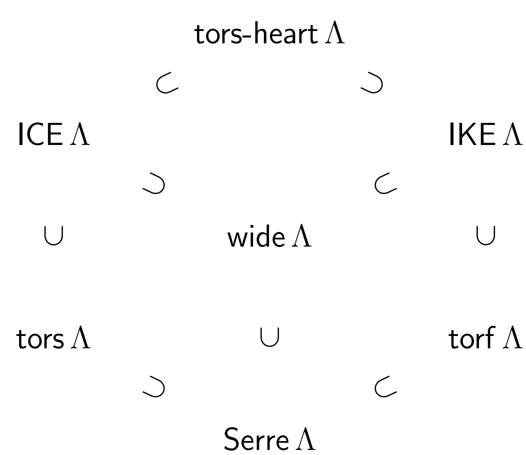
- $\mathcal{H}_{[0,\mathcal{T}]} = \mathcal{T}$.
- $\mathcal{H}_{[\mathcal{T},\mathcal{T}]} = 0$.
- $\mathcal{H}_{[\mathcal{T},\mathsf{mod}\,\Lambda]} = \mathcal{T}^{\perp}$.

By the first equality, every torsion class is a torsion heart, thus $\mathsf{tors}\,\Lambda \subseteq \mathsf{tors\text{-}heart}\,\Lambda$ holds. Similarly, by the last equality, every torsion-free class is a torsion heart, thus $\mathsf{torf}\,\Lambda \subseteq \mathsf{tors\text{-}heart}\,\Lambda$ holds.

The first result of this article is as follows.

Theorem 10 ([5])**.** *Every ICE-closed subcategory is a torsion heart. Namely, for every ICE-closed subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, there are some torsion classes $\mathcal{U}$ and $\mathcal{T}$ satisfying $\mathcal{U} \subseteq \mathcal{T}$ and $\mathcal{C} = \mathcal{H}_{[\mathcal{U},\mathcal{T}]}$.*

This shows $\mathsf{ICE}\,\Lambda \subseteq \mathsf{tors\text{-}heart}\,\Lambda$, and by duality, we also have $\mathsf{IKE}\,\Lambda \subseteq \mathsf{tors\text{-}heart}\,\Lambda$. Thus the situation looks like as follows, and all the classes of these subcategories can be obtained by $\mathsf{tors}\,\Lambda$ using the heart construction.



4. Constructing other posets from $\mathsf{tors}\,\Lambda$

In this last section, we state the main result of this article, which says that posets $\mathsf{wide}\,\Lambda$ and $\mathsf{ICE}\,\Lambda$ can be recovered from $\mathsf{tors}\,\Lambda$ by using only poset-theoretical information. Since $\mathsf{wide}\,\Lambda$ and $\mathsf{ICE}\,\Lambda$ can be regarded as full subposets of $\mathsf{tors\text{-}heart}\,\Lambda$ by Theorem 10, our strategy is to first construct $\mathsf{tors\text{-}heart}\,\Lambda$ using $\mathsf{tors}\,\Lambda$.

Definition 11. We denote $\mathsf{itv}(\mathsf{tors}\,\Lambda)$ by the set of intervals in $\mathsf{tors}\,\Lambda$, that is,

$$\mathsf{itv}(\mathsf{tors}\,\Lambda) := \{(\mathcal{U}, \mathcal{T}) \in \mathsf{tors}\,\Lambda \times \mathsf{tors}\,\Lambda \mid \mathcal{U} \subseteq \mathcal{T}\}.$$

By the definition of torsion hearts, there is a natural surjection by taking hearts:

$$\mathcal{H}_{(-)} \colon \mathsf{itv}(\mathsf{tors}\,\Lambda) \twoheadrightarrow \mathsf{tors\text{-}heart}\,\Lambda$$

Now we can state the main result of this section.

Theorem 12. *We can define an equivalence relation $\sim$ and a binary relation $\leq$ on $\mathsf{itv}(\mathsf{tors}\,\Lambda)$, which only depend on the poset structure of $\mathsf{tors}\,\Lambda$, such that $\mathcal{H}_{(-)}$ induces an isomorphism of posets:*

$$\left(\frac{\mathsf{itv}(\mathsf{tors}\,\Lambda)}{\sim}, \leq \right) \simeq (\mathsf{tors\text{-}heart}\,\Lambda, \subseteq).$$

We only give a rough explanation of the proof. It is shown in [3] that the poset $\mathsf{tors}\,\Lambda$ has the full information on *bricks*, that is, Λ-modules whose endomorphism ring is division rings, and that every torsion heart is determined by the set of bricks contained in it. The theorem now follows by describing bricks contained in the heart of intervals only by using the poset structure of $\mathsf{tors}\,\Lambda$.

By using this theorem and the characterization of intervals whose hearts are ICE-closed or wide subcategories, one obtains the following corollary.

Corollary 13. *One can recover the posets $\mathsf{ICE}\,\Lambda$ and $\mathsf{wide}\,\Lambda$ from $\mathsf{tors}\,\Lambda$ by only using the poset structure.*

This enables us to construct posets $\mathsf{ICE}\,\Lambda$ and $\mathsf{wide}\,\Lambda$ in computer if $\mathsf{tors}\,\Lambda$ is given.

References

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