GENERALIZATIONS OF THE CORRESPONDENCE BETWEEN QUASI-HEREDITARY ALGEBRAS AND DIRECTED BOCSES

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Abstract. Koenig, Külshammer and Ovsienko showed that Morita equivalence classes of quasi-hereditary algebras are in one-to-one correspondence with equivalence classes of the module categories over directed bocses. In this report, we give theorems which extend their result to Δ-filtered algebras and $\overline{\Delta}$-filtered algebras.

1. Introduction

Quasi-hereditary algebras were introduced by Cline, Parshall, and, Scott to study the highest weight categories in Lie theory [CPS]. So far, many results have been obtained for quasi-hereditary algebras. As an important fact, quasi-hereditary algebras have finite global dimensions. On the other hand, bocs theory was introduced in the context of Drozd's tame and wild dichotomy theorem and Crawley-Boevey applied it to analyze the module categories over tame algebras. The module categories over bocses behave differently from those over algebras. Koenig, Külshammer and Ovsienko connected these theories by giving equivalences between the categories of modules over directed bocses and those of Δ-filtered modules over quasi-hereditary algebras. Moreover, Brzeziński, Koenig and Külshammer showed that exact Borel subalgebras of quasi-hereditary algebras corresponding to directed bocses are homological [BKK].

In this report, we extend their results to Δ-filtered algebras and $\overline{\Delta}$-filtered algebras. Now we recall the result of [KKO] (we call this KKO theory). Their main result is as follows.

Theorem 1 ([KKO] Theorem 1.1, Corollary 1.3, [BKK] Theorem 3.13)**.** *We have a bijection*

$$\{\textit{Morita equivalent classes of quasi-hereditary algebras}\}$$
$$\updownarrow$$
$$\{\textit{Equivalence classes of the module categories over directed bocses}\}.$$

Let a quasi-hereditary algebra A and a directed bocs $\mathcal{B} = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra B.

As natural generalizations of quasi-heredity algebras, we have two classes of algebras; Δ-filtered algebras (or standardly stratified algebras) and $\overline{\Delta}$-filtered algebras. We give generalizations of KKO theory to these algebras.

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 2 ([BPS]). *We have a bijection*

$$\{\textit{Morita equivalence classes of } \Delta\textit{-filtered algebras}\}$$
$$\updownarrow$$
$$\{\textit{Equivalence classes of the module categories over weakly directed bocses}\}.$$

Let a Δ-filtered algebra A and a weakly directed bocs $\mathcal{B} = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra B.

Theorem 3 ([G]). *We have a bijection*

$$\{\textit{Morita equivalence classes of } \overline{\Delta}\textit{-filtered algebras}\}$$
$$\updownarrow$$
$$\{\textit{Equivalence classes of the module categories over one-cyclic directed bocses}\}.$$

Let a $\overline{\Delta}$-filtered algebra A and a one-cyclic directed bocs $\mathcal{B} = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological proper Borel subalgebra B.

Since Δ-filtered algebras were already studied by Bautista, Pérez, and, Salmerón in [BPS], Theorem 3 above is our new result. When we generalize KKO theory, we face some problems. We will discuss these in Section 3. The first problem concerns with the dimension of the Ext-algebra of properly standard modules. In [KKO], the directed bocs is constructed by using the Ext-algebra of standard modules over a quasi-hereditary algebra. But in general, the Ext-algebra of properly standard modules over a $\overline{\Delta}$-filtered algebra is not finite dimensional. To avoid infinite dimensional algebras, we will use a finite dimensional subspace of the Ext-algebra. This method for construction of bocses by using the subspaces is a generalization of the one used in [KKO]. The second problem is on the dimension of B of the bocs $\mathcal{B} = (B, W)$ induced from a $\overline{\Delta}$-filtered algebra. Since the Gabriel quiver of B has loops but no cycles of length more than 1, it suffices to show that each e_iBe_i is finite dimensional, which is of course equivalent to the fact that B is so.

2. Preliminaries

Throughout this report, let K be an algebraically closed field and A a finite dimensional K-algebra with n simple modules (up to isomorphisms). The category of a finitely generated left A-modules will be denoted by mod A and call its objects just A-modules. And let $K^-(\text{proj } A)$ be the upper bounded homotopy category of the category of projective A-modules. We denote simple A-modules by $S_A(i), 1 \leq i \leq n$ and corresponding projective indecomposable A-modules by $P_A(i), 1 \leq i \leq n$. But when there is not much danger of confusion, we also write $S(i), P(i)$ for $S_A(i), P_A(i)$, respectively. We write D to mean the standard K-dual $\text{Hom}_K(-, K)$. Let M be an A-module. For complexes X_* and Y_* of A-modules, write $\text{Hom}_A(X_*, Y_*)$ as the set $\Pi_{l\in\mathbb{Z}}\text{Hom}_A(X_l, Y_l)$ of sequences of A-homomorphisms (not necessary commuting to differentials of X_* and Y_*).

2.1. **Quasi hereditary, Δ-filtered, and $\overline{\Delta}$-filtered algebras.** Now, we will recall the definitions of some classes of algebras and important modules over them.

Definition 4. (1) For each $i \in \{1, \dots, n\}$, the A-modules $\Delta(i)$ and $\overline{\Delta}(i)$, called the **standard module** and the **properly standard module**, are defined by

$$\Delta(i) = \sum_{\substack{j>i \\ \varphi \in \mathrm{Hom}_A(P(i),P(j))}} \mathrm{Im}\ \varphi, \quad \overline{\Delta}(i) = \sum_{\substack{j \geq i \\ \varphi \in \mathrm{rad}_A(P(i),P(j))}} \mathrm{Im}\ \varphi,$$

respectively.

(2) We say that a module M has a **Δ-filtration**, or M is **Δ-filtered**, if there is a submodule sequence $0 = M_m \subset \cdots \subset M_1 \subset M_0 = M$ such that $M_{k-1}/M_k \cong \Delta(j)$ for some $j \in \{1, \dots, n\}$. Write $\mathcal{F}(\Delta)$ to mean the full subcategory of mod A whose objects are modules with Δ-filtrations. Similarly we define $\overline{\Delta}$-filtered modules and the category $\mathcal{F}(\overline{\Delta})$.

(3) A pair of an algebra and a total order $(A, \leq)$, or just A, is called a **Δ-filtered algebra** (resp. a **$\overline{\Delta}$-filtered algebra**) provided that every $P(i)$ has a $\{\Delta(i), \dots, \Delta(n)\}$-filtration (resp. a $\{\overline{\Delta}(i), \dots, \overline{\Delta}(n)\}$-filtration).

(4) If $(A, \leq)$ is a Δ-filtered algebra and $\Delta = \overline{\Delta}$, then it is called a **quasi-hereditary algebra**.

Definition 5. (1) An algebra B is **directed** (resp. **one-cyclic directed**) if $\mathrm{rad}_B(P_B(i), P_B(j)) = 0$ for $i \leq j$ (resp. for $i < j$).

(2) Let A be a Δ-filtered algebra. A subalgebra B of A is called an **exact Borel subalgebra** if the following conditions hold.
   - B is directed.
   - $A \otimes_B - : \mathrm{mod}\ B \to \mathrm{mod}\ A$ is exact.
   - $\Delta_A(i) \cong A \otimes_B S_B(i)$.

(3) Let A be a $\overline{\Delta}$-filtered algebra. A subalgebra B of A is called a **proper Borel subalgebra** if the following conditions hold.
   - B is one-cyclic directed.
   - $A \otimes_B - : \mathrm{mod}\ B \to \mathrm{mod}\ A$ is exact.
   - $\overline{\Delta}_A(i) \cong A \otimes_B S_B(i)$.

(4) A subalgebra B of A is **homological** if for any B-modules M, N, natural maps

$$\mathrm{Ext}^k_B(M, N) \to \mathrm{Ext}^k_A(A \otimes_B M, A \otimes_B N)$$

are epimorphisms for $k \geq 1$ and isomorphisms for $k \geq 2$.

2.2. **Bocses.** The bocs theory was introduced in Drozd's tame-wild dichotomy theorem, and Crawley-Boevey studied bocses in [C-B].

Definition 6. A bocs is $\mathcal{B} = (B, W, \varepsilon, \mu)$, or just (B, W), consisting of a finite dimensional basic K-algebra B and a B-bimodule W which has a B-coalgebra structure, that is, there exist a B-bilinear counit $\varepsilon : W \to B$ and a B-bilinear comultiplication $\mu : W \ni w \mapsto \sum w_1 \otimes w_2 \in W \otimes_B W$ (using sigma notation).

We will always assume that the counit ε of a bocs is surjective. Hereafter, let $e_1, \dots, e_n$ be pairwise orthogonal basic primitive idempotents of an algebra B.

Definition 7. (1) A bocs $\mathcal{B} = (B, W)$ is said to have a projective kernel if $\overline{W} = \operatorname{Ker} \varepsilon$ is a projective B-bimodule.

(2) A bocs $\mathcal{B} = (B, W)$ with a projective kernel is called
- **directed** if B is directed and $\overline{W} \cong \oplus_{i>j}(Be_i \otimes_K e_j B)^{d_{ij}}$,
- **weakly directed** if B is directed and $\overline{W} \cong \oplus_{i\geq j}(Be_i \otimes_K e_j B)^{d_{ij}}$,
- **one-cyclic directed** if B is one-cyclic directed and $\overline{W} \cong \oplus_{i>j}(Be_i \otimes_K e_j B)^{d_{ij}}$,

for some $d_{ij} \geq 0$, respectively.

Definition 8. The category mod $\mathcal{B}$ of finite dimensional modules over a bocs $\mathcal{B} = (B, W)$ is defined as follows:

objects: finite dimensional left B-modules
morphisms: for B-modules M and N, $\operatorname{Hom}_{\mathcal{B}}(M, N) = \operatorname{Hom}_B(W \otimes_B M, N)$
composition: for $f \in \operatorname{Hom}_{\mathcal{B}}(M, N)$ and $g \in \operatorname{Hom}_{\mathcal{B}}(N, L)$, the composition gf of f and g is given by:

$$W \otimes M \xrightarrow{\mu \otimes \mathrm{id}_M} W \otimes W \otimes M \xrightarrow{\mathrm{id}_W \otimes f} W \otimes N \xrightarrow{g} L.$$

unites: the unite morphism $\mathrm{id}_M^{\mathcal{B}} \in \operatorname{End}_{\mathcal{B}}(M)$ is given by the composition of the following maps:

$$W \otimes M \xrightarrow{\varepsilon \otimes \mathrm{id}_M} B \otimes M \xrightarrow{l_M} M,$$

where l_M is the canonical isomorphism defined by $l_M(b \otimes x) = bx$.

Definition 9 ([BB])**.** Let $\mathcal{B} = (B, W)$ be a bocs. The right Burt-Butler algebra $R = R_{\mathcal{B}}$ of the bocs $\mathcal{B}$ is defined by $\operatorname{End}_{\mathcal{B}}(B)^{\mathrm{op}}$ and whose multiplication is the composition of morphisms in mod $\mathcal{B}$ with $1_R = \mathrm{id}_B^{\mathcal{B}}$.

2.3. **A_∞-algebras and their multiplications.** In this subsection we introduce A_∞-algebras and multiplications of the algebras, refer to [Kel].

Definition 10. A $\mathbb{Z}$-graded space $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ is called an A_∞**-algebra** if there are graded linear maps $m_k : \mathcal{A}^{\otimes k} \to \mathcal{A}$ of degree $2-k$ satisfying the equalities

$$\sum_{\substack{k=r+t+u \\ r,u\geq 0, t\geq 1}} (-1)^{r+tu} m_{r+t+1}(\mathrm{id}^{\otimes r} \otimes m_t \otimes \mathrm{id}^{\otimes u}) = 0,$$

for any $k \geq 1$.

We call these maps m_k also multiplications of $\mathcal{A}$.

Remark 11 ([LPWZ]). Let $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ be a differential graded algebra with differential d of degree 1 and $Z(\mathcal{A})$, $B(\mathcal{A})$, and $H(\mathcal{A})$ be cocycles, coboundaries, and cohomology of $\mathcal{A}$, respectively. Then we have a subspace $L(\mathcal{A})$ of $\mathcal{A}$ such that

$$\mathcal{A} = Z(\mathcal{A}) \oplus L(\mathcal{A}) = B(\mathcal{A}) \oplus H(\mathcal{A}) \oplus L(\mathcal{A}).$$

Consider the graded map $G : \mathcal{A} \to \mathcal{A}$ of degree -1 satisfying $G|_{L(\mathcal{A})_k \oplus H(\mathcal{A})_k} = 0$ and $G|_{B(\mathcal{A})_k} = (d|_{L(\mathcal{A})_{k-1}})^{-1}$. Define a sequence of linear maps λ_k of degree $2-k$ as follows.

There is no map λ_1 but we define the composition $G\lambda_1$ by $-\mathrm{id}_{\mathcal{A}}$. The map λ_2 is the same as the multiplication of $\mathcal{A}$. And for $k \geq 3$, we inductively define λ_k by

$$\lambda_k = \sum_{l=1}^{k-1} \lambda_2(G\lambda_l(a^1, \dots, a^l), G\lambda_{k-l}(a^{l+1}, \dots, a^k))$$

for $a^1, \dots, a^k \in \mathcal{A}$. Let $p : \mathcal{A} \to H(\mathcal{A})$ and $i : H(\mathcal{A}) \to \mathcal{A}$ be the canonical projection and injection, respectively. Then $H(\mathcal{A})$ is an A_∞-algebra with multiplications $m_k = p\lambda_k i^{\otimes k} : H(\mathcal{A})^{\otimes k} \to H(\mathcal{A})$.

3. Relationship between $\overline{\Delta}$-filtered algebras and one-cyclic directed bocses

In this section, we will prove our main theorem which shows relationship between $\overline{\Delta}$-filtered algebras and one-cyclic directed bocses, and the detailed proofs are given in [G]. To do this, we imitate the arguments in [KKO]. We prove our main theorem by three steps. The first one is the construction of $\overline{\Delta}$-filtered algebras from one-cyclic directed bocses.

Theorem 12. *Let $\mathcal{B} = (B, W)$ be a one-cyclic directed bocs. Then its right Burt-Butler algebra R of $\mathcal{B}$ is a $\overline{\Delta}$-filtered algebra such that $\mathcal{F}(\overline{\Delta}_R) \simeq \mathrm{mod}\ \mathcal{B}$ and B is a homological proper Borel subalgebra of R.*

The second one is to consider the relationship between Morita equivalence classes of $\overline{\Delta}$-filtered algebras and equivalence classes of categories of modules with $\overline{\Delta}$-filtrations over $\overline{\Delta}$-filtered algebras.

Theorem 13 ([ADL] Theorem 2.3)**.** *Let A' be a finite dimensional algebra. Then there exists a $\overline{\Delta}$-filtered algebra A, unique up to Morita equivalence, such that the category $\mathcal{F}(\overline{\Delta}_{A'})$ and the category $\mathcal{F}(\overline{\Delta}_A)$ are equivalent. In particular, $\overline{\Delta}$-filtered algebras A and A' are Morita equivalent if and only if there exists an equivalence $F : \mathcal{F}(\overline{\Delta}_A) \to \mathcal{F}(\overline{\Delta}_{A'})$.*

The final one is the construction of one-cyclic directed bocses from $\overline{\Delta}$-filtered algebras. Difficulties for generalizing KKO theory for $\overline{\Delta}$-filtered algebras lie in the fact that properly standard modules may have self-extensions.

Remark 14. We recall the construction of bocses given in [KKO].

- Step 1 Let A be a quasi-hereditary algebra, $\mathcal{A} = \mathrm{Hom}_A(\mathbb{P}, \mathbb{P}[*]) = \oplus_{l \geq 0} \mathrm{Hom}_A(\mathbb{P}, \mathbb{P}[l])$ where $\mathbb{P}$ is a projective resolution of the direct sum of $\Delta(1), \dots, \Delta(n)$, and s the suspension defined by $(s\mathcal{A})_l = \mathcal{A}_{l+1}$. Then $H(\mathcal{A}) = \mathrm{Ext}^*_A(\Delta, \Delta)$ and it is an A_∞-algebra. Let $\{m_k\}_{k \geq 1}$ be multiplications of $H(\mathcal{A})$ as an A_∞-algebra.
- Step 2 Consider the dual maps $d_k : Q \to Q^{\otimes k}$ of $m_k s^{\otimes k}$, where $Q_l = D((sH(\mathcal{A}))_l)$.
- Step 3 Let $T[Q]$ be the tensor algebra of Q over $\oplus_{i=1}^n K\mathrm{id}_{\overline{\Delta}_i}$. Then $T[Q]$ is a differential graded algebra with differential d.
- Step 4 Put $U = T[Q]/I$, where the ideal I of $T[Q]$ is generated by $Q_{\leq -1}$ and $d(Q_{-1})$. Then the factor U is also a differential graded algebra, and is freely generated over $B = T[Q_0]/(T[Q_0] \cap I)$ by Q_1.

Step 5 Put $W = U_1/d(B)$ and take the natural epimorphism $\pi : U_1 \to W$. Consider the two homomorphisms $\mu : W \to W \otimes W$ and $\varepsilon : W \to B$ such that the following diagrams commute, respectively,

$$\begin{array}{ccccccc} U_1 & \xrightarrow{d} & U_1 \otimes U_1 & \qquad & U_1 & \xrightarrow{\cong} & (\bigoplus_i B\omega_i B) \oplus \overline{U} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi\otimes\pi} & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\varepsilon}} \\ W & \xrightarrow{\mu} & W \otimes W, & & W & \xrightarrow{\varepsilon} & B, \end{array}$$

where $\omega_i \in Q_1(i,i)$ are elements corresponding to $\mathrm{id}_{\overline{\Delta}(i)}$, and $\tilde{\varepsilon}$ maps ω_i to e_i and $\overline{U}$ to zero. Then $\mathcal{B}_A = (B, W, \mu, \varepsilon)$ is a directed bocs.

Let A be a $\overline{\Delta}$-filtered algebra. When we extend KKO theory to $\overline{\Delta}$-filtered algebra, we must face two problems. The first problem occurs in Step 1. Notice that $\mathrm{Ext}^*_A(\overline{\Delta}, \overline{\Delta})$ may be infinite dimensional although we need to take its dual in Section 2. The second problem is in Step 5. We need to check that B is finite dimensional, because its Gabriel quiver has loops.

Apply Remark 14 to $\mathcal{A} = \mathrm{Hom}_A(\mathbb{P}, \mathbb{P}[*])$ where $\mathbb{P}$ is a projective resolution of the direct sum of $\overline{\Delta}(1), \dots, \overline{\Delta}(n)$. Then we have $H(\mathcal{A}) = \mathrm{Ext}^*_A(\overline{\Delta}, \overline{\Delta}) \cong \mathrm{Hom}_{K^-(\mathrm{proj}\ A)}(\mathbb{P}, \mathbb{P}[*])$. On the first problem, in order to avoid infinite dimensional algebras, we deal with a subspace $H(\mathcal{A})_{\leq 2}$ of $H(\mathcal{A})$.

Lemma 15. *Let $\mathcal{A} = \mathrm{Hom}_A(\mathbb{P}_i, \mathbb{P}_i[*])$. Then $H(\mathcal{A})$ is an A_∞-algebra with some graded maps m_k by* Remark 14*. Consider the graded linear maps $b'_k : sH(\mathcal{A})^{\otimes k}_{\leq 1} \to sH(\mathcal{A})_{\leq 1}$ of degree -1 defined by*

$$b'_k(a^1, \dots, a^k) = \begin{cases} b_k(a^1, \dots, a^k) & \text{for } \sum_{i=1}^k |a_k| \leq 0, \\ 0 & \text{for } \sum_{i=1}^k |a_k| \geq 1, \end{cases}$$

where $|a_k|$ is the degree of a_k and $b_k : sH(\mathcal{A})^{\otimes k} \to sH(\mathcal{A})$ are the maps induced from $m_k : H(\mathcal{A})^{\otimes k} \to H(\mathcal{A})$ i.e. $sm_k = b_k s^{\otimes k}$. Then for any $k \geq 1$, we have

$$\sum_{\substack{k=r+t+u \\ r,u\geq 0, t\geq 1}} b'_{r+1+t}(\mathrm{id}^{\otimes r} \otimes b'_t \otimes \mathrm{id}^{\otimes u})\iota^{\otimes k} = 0,$$

where $\iota : sH(\mathcal{A})_{\leq 0} \to sH(\mathcal{A})_{\leq 1}$ is the canonical injection.

Proposition 16. *Take the dual statement of the result in the last lemma. Then we get the equality*

$$\sum_{\substack{k=r+t+u \\ r,u\geq 0, t\geq 1}} p^{\otimes k}(\mathrm{id}^{\otimes r} \otimes d'_t \otimes \mathrm{id}^{\otimes u})d'_{r+1+u} = 0,$$

for each $k \geq 0$, where $d'_t = D(b'_t)$ and $p : Q_{\geq -1} \to Q_{\geq 0}$ is the canonical surjection. Consider the factor algebra $T(Q_{\geq 0})/d'(Q_{\geq -1})$. Then it is a differential graded algebra with differential d' induced from the maps d'_k.

The algebra $T(Q_{\geq 0})/d'(Q_{\geq -1})$ given in above corresponds to the algebra U given in Remark 14 Step 4. So we can construct bocses by a way similar to one in Steps 4, 5.

On the second problem, in order to guarantee that B is finite dimensional, we show that e_iBe_i and $E_i = \mathrm{End}_A(\Delta(i), \Delta(i))$ are Morita equivalent for each $1 \leq i \leq n$. Let A be a $\overline{\Delta}$-filtered algebra and

$$\mathbb{P}_i : \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0$$

be a projective resolution of $\overline{\Delta}(i)$. Then P_l is a direct sum of copies of $P(i), \ldots, P(n)$ for any $l \geq 0$. And we will also write $P_l = P(i)^{\oplus c_l} \oplus \overline{P_l}$ where $\overline{P_l}$ dose not include $P(i)$ as direct summands. Let $\mathcal{A}_i = \mathrm{Hom}_A(\mathbb{P}_i, \mathbb{P}_i[*])$. Then we identify $\mathrm{Ext}^k_A(\overline{\Delta}(i), \overline{\Delta}(i))$ and $H(\mathcal{A}_i) = \mathrm{Hom}_{\mathcal{K}^-(\mathrm{proj}A)}(\mathbb{P}_i, \mathbb{P}_i[k])$.

Lemma 17. *Let A be a $\overline{\Delta}$-filtered algebra and $\mathbb{P}_i$ a projective resolution of $\overline{\Delta}(i)$. Then as a basis of $H(\mathcal{A}_i)_k$, we can choose chain maps $f = (f_l)_{l\in\mathbb{Z}} : \mathbb{P}_i \to \mathbb{P}_i[k]$ with $f_k : P_k = P(i)^{\oplus c_k} \oplus \overline{P_k} \to P(i) = P_0$ being of the form $[\pi_m, 0]$, where $1 \leq m \leq c_k$ and $\pi_m : P(i)^{\oplus c_k} \to P(i)$ is the canonical m-th projection.*

Immediately, it turns out that some chain maps $f : \mathbb{P}_i \to \mathbb{P}_i[k]$ form a basis of $B(\mathcal{A}_i)_k$ if and only if maps $f_k : P_k \to P_0$ do a basis of $\mathrm{rad}_A(P_k, P_0)$. Moreover we can take a basis of $L(\mathcal{A}_i)_{k-1}$ by choosing finitely many non-chain maps $u : \mathbb{P}_i \to \mathbb{P}_i[k-1]$ such that $u_{k-1} : P_{k-1} \to P_0$ is the zero map. Here non-chain maps mean morphisms of complexes which are not compatible with the differentials of those.

Lemma 18. *For $a^1, \ldots, a^k \in H(\mathcal{A}_i)$, we have $\lambda_k(a^1, \ldots, a^k) \in Z(\mathcal{A}_i)$.
Moreover, $\lambda_k(a^1, \ldots, a^k) \in H(\mathcal{A}_i)$ if and only if $(a^1 \circ \lambda_{k-1}(a^2, \ldots, a^k))_2$ is surjective. Further, $\lambda_k(a^1, \ldots, a^k) \in H(\mathcal{A}_i)$ for some $a^1 \in H(\mathcal{A}_i)$ if and only if for $(\lambda_{k-1}(a^2, \ldots, a^k))_2 = \sum f^j \in \mathrm{Hom}_A(P_2, P(i))^{\oplus c_1} \oplus \mathrm{Hom}_A(P_2, \overline{P_1})$, there exists $f^j \in \mathrm{Hom}_A(P_2, P(i))$ is surjective.*

Now put $A' = A/eAe$ where $e = \sum_{k=i+1}^n e^A_k$ and $Ae^A_k \cong P_A(k)$. Then A' is also a $\overline{\Delta}$-filtered algebra and $\overline{\Delta}(i)$ is an A'-module. Let $\mathcal{A}'_i = \mathrm{Hom}_{A'}(\mathbb{P}'_i, \mathbb{P}'_i)$ where $\mathbb{P}'_i$ is a projective resolution of $\overline{\Delta}$ as an A'-module. Applying Lemma 17 to A', we can conclude that $H(\mathcal{A}_i)$ and $H(\mathcal{A}'_i)$ are isomorphic as algebras because $\dim H(\mathcal{A}_i)_k = c_k = \dim H(\mathcal{A}'_i)_k$ and their multiplications are compositions of chain maps. Since we can similarly argue in all these stories without using $P(i+1), \ldots, P(n)$, we conclude that the following proposition holds.

Proposition 19. *$H(\mathcal{A}_i)$ and $H(\mathcal{A}'_i)$ are also isomorphic as A_∞-algebras.*

By the construction of B in Remark 14 Step 4 and Keller's reconstruction theorem, we get the following.

Theorem 20. *e_iBe_i and E_i are Morita equivalent.*

To show our main theorem, we finally need the following.

Theorem 21. *Let A be a $\overline{\Delta}$-filtered algebra. Then the bocs $\mathcal{B}_A = (B, W)$ constructed above is one-cyclic directed and satisfies* $\mathrm{mod}\, \mathcal{B}_A \simeq \mathcal{F}(\overline{\Delta}_A)$.

Combine Theorems 12, 13, and 21, then we get Theorem 3.

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