POSITIVE CLUSTER COMPLEXES AND τ-TILTING SIMPLICIAL COMPLEXES

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Abstract. Gabriel's theorem, shown in 1972, is a theorem that classifies path algebras of finite representation type using Dynkin diagrams, and is a very important theorem that suggests a connection between Lie theory and the representation theory of algebras. In this paper, I will generalize Gabriel's theorem by using cluster algebra theory, which has been rapidly developed recently and is closely related to both Lie theory and the representation theory of algebras.

1. Introduction

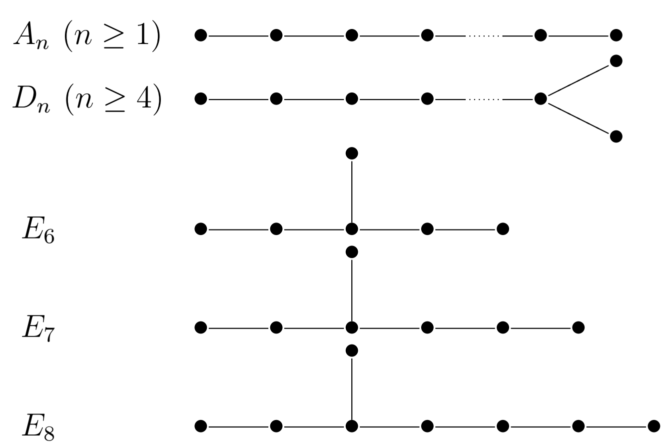
In this paper, we explain the results of the paper [9] from the viewpoint of representation theory of algebras. In particular, we consider an extension of Gabriel's theorem, which is one of the results in the representation theory of algebras, using cluster algebra theory.

Theorem 1 ([8]). *Let K be a field.*

(1) When Q is a connected quiver, It is an equivalence that the path algebra KQ is of finite representation type and that Q is a quiver of type A, D, E.

(2) If Q is a quiver of type A, D, E, then the number of isomorphic classes of indecomposable modules of KQ does not depend on the orientation of the arrow of Q, but only on its underlying graph.

Here, a graph of type A, D, E is one of the graphs in the following figure, and a quiver of type A, D, E is one of these graphs with an orientation on each edge.



The detailed version of this paper will be submitted for publication elsewhere.

This theorem was proved by Gabriel in 1972, and then Bernstein, Gel'fand, Ponomalev [3] gave another proof using the method of comparing the module categories of KQ and KQ' with a quiver Q' which swaps all the directions of the arrows going into or out of one sink or source vertex of Q. In this paper, I will explain the following theorem, which extends this theorem.

Theorem 2 ([9])**.** *Let K be a algebraically closed field.*

(1) When Q is a connected quiver, it is an equivalence that the cluster-tilted algebra KQ/I is of finite representation type and that Q is a mutation equivalence in any quiver of type A, D, E.

(2) Let $\Lambda = KQ/I$ and $\Lambda' = KQ'/I$ be cluster-tilted algebra of finite representation type. If Q and Q' are shifted by a sink or source mutation, then for any $k \in \mathbb{N}$, the number of isomorphism classes of basic τ-rigid modules of Λ and that of basic τ-rigid modules of Λ' such that there are k indecomposable factors are equal.

The definition of cluster-tilted algebras, mutation equivalences, basic τ-rigid modules, etc., will be defined in later sections. Restricting to the case where Q is a tree in Theorem 2, the cluster-tilted algebra is a path algebra, and by imposing $K = 1$, the basic τ-rigid module coincides with the indecomposable module. In addition, by imposing $K = 1$, the basic τ-rigid module coincides with the indecomposable module, so we state here that Theorem 2 is given as a specialization of Theorem 1 (where K is algebraically closed). Moreover, Theorem 2 can actually be obtained as a corollary of stronger theorem. In this section, we will briefly discuss the approach we will use to prove these theorems.

A basic τ-rigid module on the algebra Λ is a module given by a direct sum of indecomposable modules that satisfy certain conditions. We consider these moduless as simplices, and collect all of these simplices to form a simplicial complex. This is called the *τ-tilting simplicial complex* of Λ. Detailed definitions will be given in later sections. They are simplicial complexes that have been studied by [12, 11], especially when the algebra is hereditary. The τ-tilting simplicial complex was later found by [1] to be a subcomplex of a more symmetric simplicial complex, the *support τ-simplicial complex.* It is known that this support τ-tilting simplicial complex coincides with the *cluster complex* in the cluster algebra theory, and the τ-tilting simplicial complex corresponds to what is called the *positive cluster complex* in the cluster algebra side. Therefore, information such as the number of τ-rigid modules is attributed to the number of simplices in the corresponding positive cluster complex. The approach we take in this paper is to investigate the number of modules in the algebra from the viewpoint of cluster algebra theory. In the field of the representation theory of algebra, if we change the algebra to be considered from the algebra KQ given by Q to KQ' given by Q', the modules given by the algebra become completely different, and it is difficult to compare the τ-tilting simplicial complexes. On the other hand, if we use cluster algebra theory to describe the two τ-tilting simplicial complexes, and if there is a special relation between Q and Q', these simplicial complexes can be viewed as different subcomplexes of one cluster complex, and comparison is very easy. In fact, this is a more general view of the category equivalence of subcategory of the module category given by [3], and this comparison is the essence of Theorem 2.

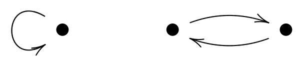
In the next section, we will discuss the basics of cluster algebra theory and τ-tilting theory, and then we will discuss these approaches in more detail.

2. Basics on cluster algebra

In cluster algebra theory, a pair of variables and a quiver, called a seed, and an operation called mutation, which creates a new seed from it, play an important role. We begin with introducing them.

Definition 3 ([7, Definition 2.3])**.** Let $\mathcal{F}$ be a rational function field with n variables. We define the *labeled seed* of $\mathcal{F}$ as a pair $(\mathbf{x}, Q)$ satisfying the following conditions.

- $\mathbf{x} = (x_1, \dots, x_n)$ is a tuple of algebraically independent n variables which is free generating set of $\mathcal{F}$.
- Q has n vertices and does not have loops and 2-cycles (see below figures).



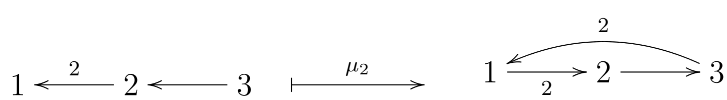
$\mathbf{x}$ is called (labeled) *cluster*, and its elements are called *cluster variables*.

From now on, each of the n vertices will be associated with a natural number between 1 and n.

Definition 4. Let Q be a quiver and take its vertex j. We define the *quiver mutation* $\mu_j(Q)$ in the direction j by the following procedure using Q.

(1) We reverse all the arrows going in and out of j.
(2) for each pair of vertices (i, k) with arrows entering and exiting j and the arrows between them $i \xrightarrow{b_{ij}} j \xrightarrow{b_{jk}} k$, we add $i \xleftarrow{b_{ij}b_{jk}} k$.
(3) we remove all 2-cycles.

We give an example of a quiver mutation:



From definition, if the mutation is performed at a source or sink vertex, the steps after (2) are skipped, resulting in an operation that replaces all arrows around the vertex used for the mutation. In other words, mutation is a generalization of the operation used in [3] to replace all arrows entering and leaving a single sink or source vertex.

Definition 5. Take a seed $(Q, \mathbf{x} = (x_1, \dots, x_n))$ and take one vertex j of Q. We define the *cluster mutation*$\mu_j(\mathbf{x})$ in the direction j using $(\mathbf{x}, Q)$ as follows:

$$x'_i = \begin{cases} x_i & \text{if } i \neq j \\ \dfrac{\prod\limits_{1 \leq i \leq n} x_i^{\max(0, b_{ij})} + \prod\limits_{1 \leq i \leq n} x_k^{\max(0, b_{ji})}}{x_j} & \text{if } i = j. \end{cases} \tag{2.1}$$

We give a cluster mutation. When a seed is a pair of $1 \xleftarrow{2} 2 \longleftarrow 3$ and $\mathbf{x} = (x_1, x_2, x_3)$, we have

$$\mu_2(\mathbf{x}) = \left(x_1, \frac{x_1^2 + x_3}{x_2}, x_3 \right).$$

By using them, we define a seed mutation:

Definition 6. Take a seed $(Q, \mathbf{x})$ and take one vertex j of Q. The *seed mutation* $\mu_j(\mathbf{x}, Q)$ in the direction j is defined by using $(\mathbf{x}, Q)$ as follows.

$$\mu_j(\mathbf{x}, Q) = (\mu_j(\mathbf{x}), \mu_j(Q)). \tag{2.2}$$

By combining the previous examples, we have a seed mutation

$$\begin{array}{ccc} 1 \xleftarrow{\ 2\ } 2 \longleftarrow 3 & \xmapsto{\ \mu_2\ } & 1 \xrightarrow[2]{} 2 \longrightarrow 3 \quad (3 \xrightarrow{\ 2\ } 1) \\ (x_1, x_2, x_3) & \xmapsto{\ \mu_2\ } & \left(x_1, \dfrac{x_1^2 + x_3}{x_2}, x_3 \right). \end{array}$$

Then μ_k is a congruence, i.e., $\mu_k \circ \mu_k(\mathbf{x}, Q) = (\mathbf{x}, Q)$, which is confirmed by direct calculation. From here, we know that $\mu_k(\mathbf{x}, Q) = (\mathbf{x}', Q')$ is the seed.

Let $\mathbb{T}_n$ be a graph with n edges extending from an arbitrary vertex, and each edge labeled with $1, \ldots, n$. Let n edges from a single vertex be labeled differently. This graph $\mathbb{T}_n$ is called *n-regular tree.* When $t, t' \in \mathbb{T}_n$ is connected by an edge labeled with ℓ, we denote it as $t \overset{\ell}{\text{---}} t'$.

Definition 7 ([7, Definition 2.9])**.** A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (\mathbf{x}_t, B_t)$ to every vertex $t \in \mathbb{T}_n$ such that the labeled seeds Σ_t and $\Sigma_{t'}$ assigned to the endpoints of any edge $t \overset{k}{\text{---}} t'$ are obtained from each other by the seed mutation in direction k. We denote by $P\colon t \mapsto \Sigma_t$ this assigment. The elements of $\mathbf{x}_t$ are denoted as follows:

$$\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}). \tag{2.3}$$

Here, in order to construct the above cluster pattern, we can choose one vertex t_0 of $\mathbb{T}_n$ and assign one seed $(\mathbf{x}, Q)$ to it, and then inductively determine the correspondence between the vertex and the seed of $\mathbb{T}_n$ using the correspondence between the mutation and the edge of $\mathbb{T}_n$. We call the seed Σ_{t_0} associated with this vertex t_0 the *initial seed.* The cluster algebra is defined as the subalgebra generated by the cluster variables contained in this seed.

Definition 8 ([7, Definition 2.11])**.** Given an arbitrary initial seed $(\mathbf{x}, Q)$ and its cluster pattern,

$$\mathcal{X}(Q) = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i;t} : t \in \mathbb{T}_n, 1 \leq i \leq n\}, \tag{2.4}$$

is defined as the union set of all the cluster variables appearing in the cluster pattern. Furthermore, the *cluster algebra* $\mathcal{A}(Q)$ associated with a given cluster pattern is defined as the $\mathbb{Z}$-subalgebra of $\mathcal{F}$ generated by all the cluster variables, i.e., $\mathcal{A}(Q) = \mathbb{Z}[\mathcal{X}(Q)]$.

We will focus on the case where $\mathcal{X}(Q)$ is a finite set.

Definition 9. When $\mathcal{X}(Q)$ is a finite set, we say that the cluster algebra $\mathcal{A}(Q)$ or its cluster pattern is of *finite type.*

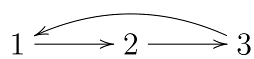
The classification of finite types has already been done, and the results are very compatible with Lie theory and others. Before stating the classification theorem, let us define the mutation equivalence of a quiver.

Definition 10. For a quiver Q and Q', if there exists a sequence of mutations $\mu_{i_1}, ..., \mu_{i_m}$ such that $Q' = \mu_{i_m} \circ \cdots \circ \mu_{i_1}(Q)$, then Q and Q' are said to be *mutation equivalence.*

The following theorem is a classification theorem of finite type.

Theorem 11 ([6, Theorem 1.8])**.** *An irreducible cluster algebra (a cluster algebra that cannot be described by a direct product of two cluster algebras) $\mathcal{A}(Q)$ is of finite type if and only if Q is a mutation equivalent to any quiver of type A, D, E.*

Note that two quivers of type A, D, E that have the same graph (differing only in the direction of the arrow) are mutation equivalent. Also, some quivers that are mutation equivalent to a quiver of type A, D, E do not have a graph of type A, D, E. For example, the following quiver



is mutation equivalent to a quiver of type A_3.

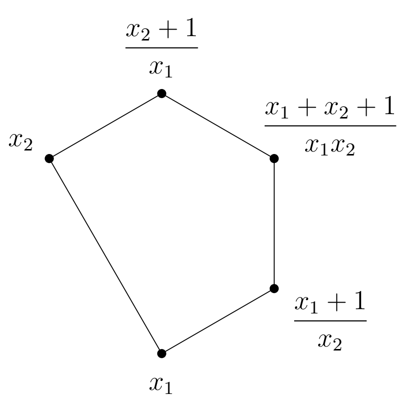
Next, we will discuss cluster complexes and positive cluster complexes.

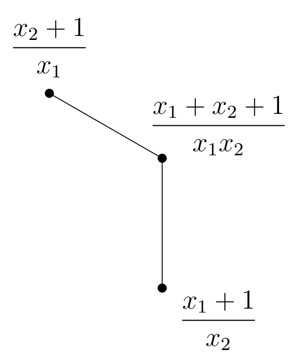
Definition 12.
- A simplicial complex whose vertex set is $\mathcal{X}(Q)$ and simplicial set consists of subsets of cluster is called a *cluster complex* and is denoted by $\Delta(Q)$.
- The full subcomplex of a cluster complex $\Delta(Q)$ with the vertices corresponding to the initial variables $\{x_1, \ldots, x_n\}$ removed is called a *positive cluster complex* and is denoted by $\Delta^+(Q)$.

An example of $Q = 1 \leftarrow 2$ is given below. First, let $\mathbb{T}_2$ be the following tree:

$$\cdots \overset{1}{—} t_0 \overset{2}{—} t_1 \overset{1}{—} t_2 \overset{2}{—} t_3 \overset{1}{—} t_4 \overset{2}{—} t_5 \overset{1}{—} \cdots .$$

Seeds are assigned to this tree in order starting from t_0. The seed is all of those listed in Table 1 below at $0 \leq t \leq 4$. To be precise, there are some clusters where the order of the cluster variables is reversed (e.g. $t = 5$), but we consider clusters with the same cluster variables to be identical. The cluster complex and the positive cluster complex are as follows.





t	Q_t	$\mathbf{x}_t$	
0	$1 \leftarrow 2$	x_1	x_2
1	$1 \rightarrow 2$	x_1	$\frac{x_1+1}{x_2}$
2	$1 \leftarrow 2$	$\frac{x_1+x_2+1}{x_1x_2}$	$\frac{x_1+1}{x_2}$
3	$1 \rightarrow 2$	$\frac{x_1+x_2+1}{x_1x_2}$	$\frac{x_2+1}{x_1}$
4	$1 \leftarrow 2$	x_2	$\frac{x_2+1}{x_1}$
5	$1 \rightarrow 2$	x_2	x_1

TABLE 1. Seeds

Clearly, the fact that the cluster algebra $\mathcal{A}(Q)$ is of finite type is equivalent to the fact that $\Delta(Q), \Delta^+(Q)$ is finite as a simplicial complex. The seed mutation can be regarded as a transfer from the maximal simplex corresponding to the cluster $\mathbf{x}$ to another maximal simplex where one vertex is replaced by a different vertex, just like the transfer from one maximal simplex to the "neighboring" maximal simplex in a cluster complex or positive cluster complex. Furthermore, if Q and Q' are mutation equivalent, then $\Delta(Q) \simeq \Delta(Q')$. Indeed, if Q and Q' are mutation equivalent, then $\Delta(Q)$ with $(\mathbf{x}, Q)$ as the starting point always has a maximal simplex corresponding to the cluster $\mathbf{x}'$ of $(\mathbf{x}', Q')$. If we see the cluster complex $\Delta(Q)$ with $(\mathbf{x}', Q')$ as the initial seed, it is nothing but $\Delta(Q')$ by definition. In other words, if Q and Q' are mutation equivalent, then $\Delta(Q)$ and $\Delta(Q')$ are different in terms of where they start in the cluster complex, but as complexes, they are exactly the same. In light of this, th epositive cluster complex $\Delta^+(Q), \Delta^+(Q')$ can be viewed as a simplicial complex consisting of the same cluster complex $\Delta(Q)$ with another part removed. This is a point of view not found in the τ-tilting simplicial complex defined in the next section.

3. BASICS ON τ-TILTING THEORY AND CLUSTER-TILTED ALGEBRA

3.1. **τ-tilting theory.** In this section, we mainly summarize the basics in the τ-tilting theory given by [1]. From this section on, K is assumed to be an algebraic closed. Let Λ be a finite dimensional K-algebra, and denote its finitely generated module category by $\mathsf{mod}\,\Lambda$, and the full subcategory of $\mathsf{mod}\,\Lambda$ by projective modules by $\mathsf{proj}\,\Lambda$.

Consider the following special module in Λ. Let $M \in \mathsf{mod}\,\Lambda$ be a module satisfying $\mathsf{Hom}_{\mathcal{C}}(M, \tau M) = 0$. Here, τ is the AR-translation in Λ. Let $|M|$ be the number of indecomposable factors of M that are nonisomorphic to each other. If the τ-rigid module

M satisfies $|M| = |\Lambda|$, then M is said to be a *τ-rigid module*. Next, we give a generalized notion of τ-rigid modules and τ-tilting modules.

Definition 13. Let (M, P) be a pair satisfying $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$.

(1) (M, P) is called a *τ-rigid pair* of Λ when M is τ-rigid and $\mathsf{Hom}_\Lambda(P, M) = 0$,

(2) (M, P) is called a *τ-tilting pair* of Λ when (M, P) is τ-rigid pair satisfying $|M| + |P| = |\Lambda|$.

The τ-tilting pair (M, P) is *directly irreducible* (*basic*) if and only if $M \oplus P$ is indecomposable (basic). Note that when (M, P) is indecomposable, either M or P is zero. Using this, we define a simplicial complex.

Definition 14.

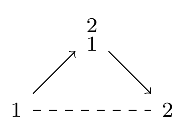
(1) We define *support τ-tilting simplicial complex* $\Delta(s\Lambda)$ of Λ as a simplicial complex whose vertex set consists of isomorphism classes of indecomposable τ-rigid pairs of Λ and whose simplicial set comsist of isomorphism classes of basic τ-rigid pairs.

(2) We define *τ-tilting simplicial complex* $\Delta(\Lambda)$ of Λ as a simplicial complex whose vertex set consists of isomorphism classes of indecomposable τ-rigid modules of Λ and whose simplicial set consist of isomorphism classes of the basic τ-rigid modules.

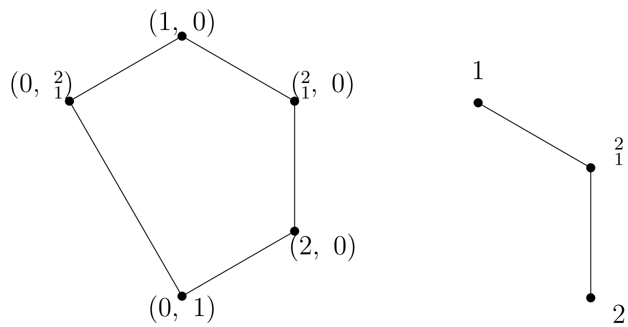
Note that any finite dimensional basic K-algebra Λ is multiplied by some (unique) connected quiver Q_Λ and an admissible ideal I in the form KQ_Λ/I. We define a τ-tilting mutation corresponding to the seed mutation of the cluster algebra.

Proposition 15 ([1, Theorem 2.18])**.** *Let Λ be a finite dimensional K-algebra and $(M \oplus N, P \oplus Q)$ be a τ-rigid pair of Λ. Let (N, Q) be an indecomposable τ-rigid pair. There exists the unique indecomposable τ-rigid pair (N', Q') such that $(M \oplus N', P \oplus Q')$ is a τ-tilting pair. Then, $(M \oplus N', P \oplus Q')$ is called the* τ-tilting mutation *of (M, P) in the direction (N, Q), and is denoted by $(M \oplus N', P \oplus Q') = \mu_{(N,P)}(M \oplus N, P \oplus Q)$.*

In $\Delta(s\Lambda)$ and $\Delta(\Lambda)$, the τ-tilting mutation can be viewed as an operation that moves from one maximal simplex to the next maximal simplex. This is similar to the relationship between seed mutations, cluster complexes, and positive cluster complexes. The following is an example of $\Lambda = K(1 \leftarrow 2)$ for a support τ-tilting simplicial complex and a τ-tilting simplicial complex. The AR-quiver of $\mathsf{mod}\,K(1 \leftarrow 2)$ is as follows.



Here, the number of numbers i in each component represents the dimension of the linear space on each vertex i in the representation of $1 \leftarrow 2$. For example, $\substack{2\\1}$ represents a representation (or the corresponding module) in which each vertex $1, 2$ has a linear space of dimension 1, and the linear map between them is an identity map. In this complex, for example, $(M, P) = (1 \oplus \substack{2\\1}, 0)$ is a τ-tilting pair. Also, the mutation in direction $(\substack{2\\1}, 0)$ of this pair is $(M', P') = (1, \substack{2\\1})$. Based on these calculations, support τ-tilting simplicial complex and τ-tilting complex of $\Lambda = K(1 \leftarrow 2)$ can be constructed as follows.



These simplicial complexes are consistent with the example of $Q = 1 \leftarrow 2$ in the cluster and positive cluster complexes seen in section 2. We will be discussed in detail for it in Section 4. When $\Delta(s\Lambda)$ is finite as a simplicial complex, we say that Λ is a *τ-tilting finite type*. Clearly, this is equivalent to saying that $\Delta(\Lambda)$ is finite as a simplicial complex.

3.2. **Cluster-tilted algebra.** Let Q be acyclic quiver. An acyclic quiver is a quiver without an oriented cycle. Consider the category $\mathcal{C}_Q = \mathcal{D}^b(KQ)/\tau^{-1}[1]$. This category is the orbital category of bounded derivatived category $\mathcal{D}^b(kQ)$ by $\tau^{-1}[1]$, and such a category is called the *cluster sphere*. For an additive category $\mathcal{C}$, when $T \in \mathcal{C}$ satisfies the following conditions, T is called the *cluster tilting object* of $\mathcal{C}$.

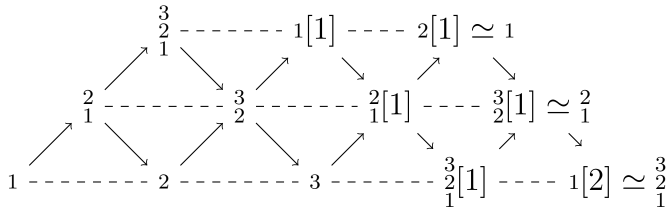
- $\mathsf{Hom}_{\mathcal{C}}(T, T[1]) = 0$,
- $\mathsf{Hom}_{\mathcal{C}}(T, Y[1]) = 0$ implies $Y \in \mathsf{add}\, T$,

where $\mathsf{add}\, T$ is a full subcategory of $\mathcal{C}$ by direct summands of direct sum of T. Mutation operations are also defined on cluster tilting objects, such as a seed of a cluster algebra or a τ-tilting module in τ-tilting theory.

Proposition 16 ([10, Theorem 5.3])**.** *Let $\mathcal{C}_Q$ be thecluster category and $T = U \oplus X$ be a cluster tilting object of $\mathcal{C}_Q$. Let X be an indecomposable object. There exists an indecomposable object X' such that $T = U \oplus X'$ is a cluster tilting object. The object $T \oplus U'$ is called the* mutation of $T \oplus U$ in direction U*, and is denoted by $T \oplus U' = \mu_U(T \oplus U)$.*

We consider constructing an algebra from cluster tilting objects. A finite dimensional algebra Λ is said to be a cluster-tilted algebra if there exists a cluster category $\mathcal{C}_Q$ and a cluster tilting object T such that $\Lambda \cong \mathsf{End}_{\mathcal{C}_Q} T^{\mathrm{op}}$. In this case, it is known that Q and I, which give $\Lambda = KQ_\Lambda/I$, are uniquely determined for Λ [5, Theorem 2.3]. Also, the cluster-tilted algebra coincides with the path algebra if Q is acyclic. Let T, T' be the cluster tilting object of $\mathcal{C}_Q$ and let $\Lambda = \mathsf{End}_{\mathcal{C}_Q} T^{\mathrm{op}}, \Lambda' = \mathsf{End}_{\mathcal{C}_Q} T'^{\mathrm{op}}$. When T' is obtained from T by cluster tilting mutation, Q'_Λ is obtained from Q_Λ by quiver mutation [4, Theorem I.1,6]. When T is a maximal basic projective object, then $Q_\Lambda = Q$. Thus, if Λ is an endomorphism algebra of a cluster tilting object in $\mathcal{C}_Q$, then Q_Λ is a quiver which is mutation equivalent to Q.

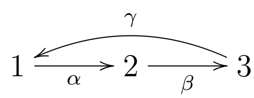
The following is an example of $Q = 1 \leftarrow 2 \leftarrow 3$. The AR-quiver is as follows.



For example, $T = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ is a cluster tilting object, and mutating it in the direction $\begin{smallmatrix}2\\1\end{smallmatrix}$ gives $T' = 1 \oplus 3 \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$. In this case, $\mathsf{End}_{\mathcal{C}_Q} T^{\mathrm{op}} = KQ$, and

$$\mathsf{End}_{\mathcal{C}_Q} T'^{\mathrm{op}} = KQ'/\langle \beta\alpha, \gamma\beta, \alpha\gamma \rangle,$$

where Q' is the following quiver:



(α, β, γ are the names of the respective arrows). Moreover, Q' is a quiver mutation of Q in direction 2.

4. Identification of positive cluster complexes and τ-tilting simplicial complexes

It is known that the simplicial complexes introduced in the previous two sections, the support τ-tilting simplicial complexes, and the positive cluster complexes and τ-tilting simplicial complexes are the same as simplicial complexes..

Theorem 17 ([9, Theorem 6.6])**.** *We have* $\Delta(s\Lambda) \simeq \Delta(Q_\Lambda)$ *and* $\Delta(\Lambda) \simeq \Delta^+(Q_\Lambda)$

Therefore, by using the classification theorem for finite types of cluster algebras (Theorem 11), we obtain Theorem 2 (1), which is the main theorem of this paper.

Theorem 18 (reprint)**.** *When Q is a connected quiver, it is an equivalence that the cluster-tilted algebra KQ/I is of finite representation type and that Q is a mutation equivalence in any quiver of type A, D, E.*

Note that in the cluster-tilted algebra, being of τ-tilting finite type is equivalent to being of finite representation type (see, for example, [13]), which verifies the above theorem.

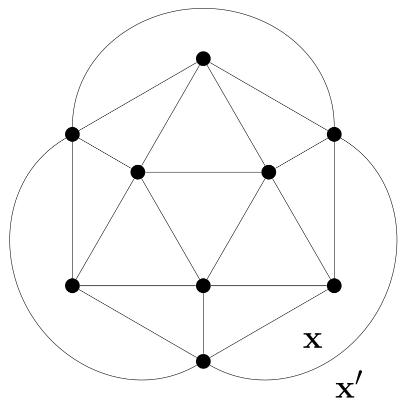
5. Reduction theorem on the difference of face vectors of cluster complexes

An important invariant in simplicial complex is the *face vector* (f-vector). This is a vector whose entries are the numbers of sinplices in each dimension of the simplicial complex. For a finite simplicia; complex Δ, we define it as

$$f(\Delta) = (f_{-1}, \dots, f_k),$$

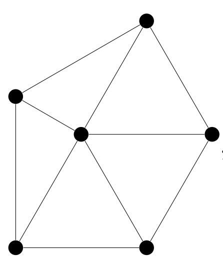
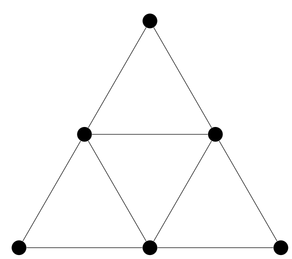
where f_i is the number of i-dimensional simplices in Δ, and the empty set is regarded as a -1-dimensional simplex. We consider the face vector of the positive cluster complex. In

[3], they considered the change of module categorys before and after quiver mutation at the source and sink vertex, so we will generalize this to consider the change in the positive cluster complexes determined by the quiver before and after mutation. A cluster complex of type A_3 is a simplicial complex as follows (Note that the term "cluster complex of type A_3" is well-defined, since all cluster complexes formed from a quiver of type A_3 and a quiver mutation equivalent to it are isomorphic). Here, note this simplicial complex is embedded in 3-dimension space and is cut open and expanded in the 2-dimension plane, and the outermost three vertices also form a simplex.



In this case, the face vector is $f(\Delta(Q)) = (1, 9, 21, 14)$.

When $Q = 1 \leftarrow 2 \leftarrow 3$, the simplex corresponding to the cluster $\mathbf{x}$ in the seed $(Q, \mathbf{x})$ containing this quiver corresponds to the simplex surrounding $\mathbf{x}$ in the above figure. The simplex corresponding to the cluster $\mathbf{x}'$ in the seed $(\mathbf{x}', Q')$ obtained by mutation at vertex 2 corresponds to the simplex surrounding $\mathbf{x}$ in the above figure. The positive complex $\Delta^+(Q)$ was obtained by removing from $\Delta(Q)$ all the vertices corresponding to all the cluster variables in the cluster $\mathbf{x}$ of $(\mathbf{x}, Q)$ and all the simplices containing those vertices. Therefore, $\Delta^+(Q)$,$\Delta^+(Q')$ are the simplicial complexes as follows.

(5.1) $\Delta^+(Q) =$  , $\Delta^+(Q') =$ 

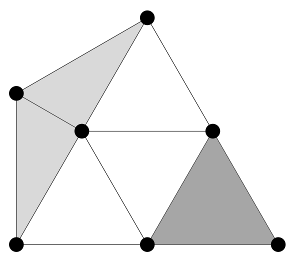
In this case, each face vector is $f(\Delta^+(Q)) = (1, 6, 10, 5), f(\Delta^+(Q')) = (1, 6, 9, 4)$. Now consider the difference $(0, 0, 1, 1)$ between the two vectors. The value of this difference is the difference of the face vectors of the τ-tilting simplicial complexes $\Delta(\Lambda), \Delta(\Lambda')$ in the algebras $\Lambda = KQ, \Lambda' = KQ'/I$ in the τ-tilting theory (see Theorem 17). Furthermore, if we look at this vector component by component, we can see that it is exactly the difference between the number of isomorphic classes of the basic τ-rigid with $k(\in \mathbb{Z}_{\geq 0})$ components in Λ and that in Λ'. In other words, when the difference of these vectors is a 0-vector, we are in the situation described by the claim of Theorem 2 (2). Therefore, all we have to do is to think about the conditions under which the mutation between Q and Q' must satisfy for the difference to be a 0 vector. An important theorem to understand this is the following theorem.

Theorem 19 ([9, Theorem 3.5])**.** *Let $\mathcal{A}(Q)$ be a cluster algebra of finite type. If Q becomes Q' by quiver mutation in direction k, then we have*

$$f(\Delta^+(Q)) - f(\Delta^+(Q')) = \left[f(\Delta^+(Q' - \{k\}))\right]_1 - \left[f(\Delta^+(Q - \{k\}))\right]_1 ,$$

where $Q - \{k\}$ is a full subquiver consisting of vertices other than k of Q, and $[v]_1$ is a vector with zero inserted in the first component and the components of v shifted by one to the right.

We will see more about the meaning of the formula in this theorem, again using an example of type A_3. When we consider the difference between two simplicial complexes in (5.1), we do not need to consider the common part. In other words, what is important is the difference of the face vectors of the simplicial complexes in the shaded area with different colors in the figure below.

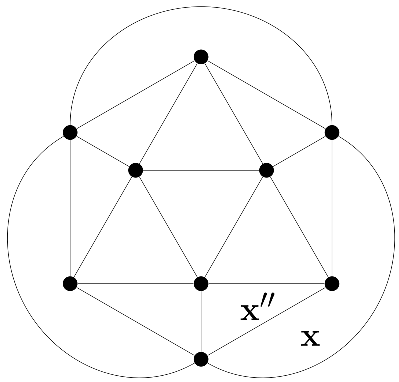


The two simplicial complexes can actually be described using a quiver of positive cluster complexes with the 2 vertex used for mutation removed. The lighter shading is a cone with $\Delta^+(Q' - \{2\}) = \Delta^+(1 \leftarrow 3)$ as its base, and the darker one is a cone with $\Delta^+(Q - \{2\}) = \Delta^+(1 \quad 3)$ as its base. We know that this fact holds not only in this specific case, but in all the situations considered in the theorem. In this way, the difference of the face vectors of a positive cluster complex can be reduced to the difference of the face vectors of positive cluster complexes of a smaller quiver. The above considerations lead to the equality in Theorem 19.

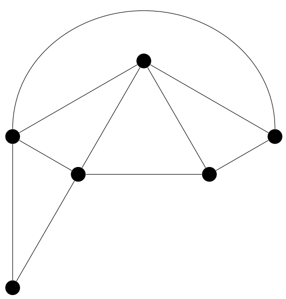
Now, the important point in Theorem 19 is that the amount of increase or decrease of the face vector of the positive cluster complex before and after mutation depends on the information of the parts of the quiver other than the vertex used for mutation. In other words, if there is no change in the arrows other than those around vertex k of the quiver before and after the mutation, the value of the face vector will not change. In particular, the mutation at the source and sink points satisfies this condition. From the above, Theorem 2 (2) is derived.

Theorem 20 (Reprint)**.** *Let $\Lambda = KQ/I$ and $\Lambda' = KQ'/I$ be cluster-tilted algebra of finite representation type. If Q and Q' are shifted by a sink or source mutation, then for any $k \in \mathbb{N}$, the number of isomorphism classes of basic τ-rigid modules of Λ and that of basic τ-rigid modules of Λ' such that there are k indecomposable factors are equal.*

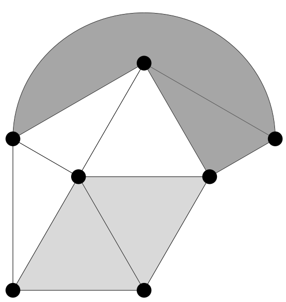
The previous example was an example where the face vectors before and after the mutation do not match, so let's look at an example where they do match. Let $(Q'', \mathbf{x}'')$ be a seed of the mutation at vertex 1 from $(Q, \mathbf{x})$ with $Q = 1 \leftarrow 2 \leftarrow 3$. Since this is a mutation at the sink vertex, the face vectors should match from the previous theorem. The maximal simplex corresponding to $\mathbf{x}''$ is the position shown in the figure below.



Then we have

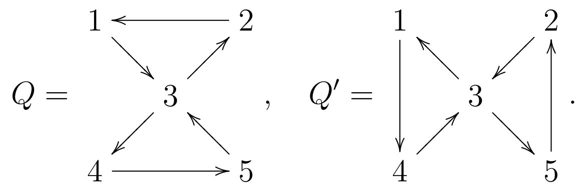
$\Delta^+(Q'') =$ 

and $f(\Delta^+(Q'')) = (1, 6, 10, 5) = f(\Delta^+(Q))$. The simplicial complexes exchanged before and after the mutation are shown in the shaded area below.



We can see that both the light and dark cones have $\Delta^+(Q - \{1\}) = \Delta^+(Q'' - \{1\}) = \Delta^+(2 \leftarrow 3)$ as their base. Thus, when mutating at a sink or source vertex, the part of the quiver that is removed and the part that is added are isomorphic as a simplicial complex (note that $\Delta^+(Q)$ and $\Delta^+(Q'')$ are not isomorphic). This can be seen as an analogue or generalization of the tilting theorem of Brenner-Butler [2] applied to APR-tilting modules (i.e., the mirror reflection property in [3]) in the representation theory of algebras. If we see that the torsion class of KQ have the common part shown in white, the torsion-free class of KQ'' have the light gray part, the torsion-free class of KQ have the dark gray part, and the torsion-free class of KQ'' have the white part, we can see that the common white part means that the torsion class of KQ is a category equivalent to the rorsion-free classes of KQ'', and the fact that the gray parts are isomorphic corresponds to the fact that the torsion-free class of KQ is category equivalent to the torsion class of KQ''.

The assumption of the theorem 20 is not a necessary condition, and there are mutations that do not change the face vector even if they are not at the source or sink vertex. For example, the following quiver pair satisfies this condition:



The two quivers Q, Q' are mutually transferred by the quiver mutation in the direction 3, and obviously vertex 3 is neither a source nor a sink. Also, since this quiver is a mutation equivalent to an A_5-type quiver, the cluster complex is of finite type. Since both Q and Q' are of type $A_2 \times A_2$ when k is removed, the difference between the face vectors of $\Delta^+(Q)$ and $\Delta^+(Q')$ is a 0-vector from Theorem 19 (actually a little stronger, since Q and Q' are isomorphic as a quiver, we have $\Delta^+(Q) \simeq \Delta^+(Q')$).

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