THE STRUCTURE OF ADAMS GRADED DG ALGEBRAS AND COHEN-MACAULAY REPRESENTATIONS

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Abstract. We study tilting and cluster tilting theory for graded Gorenstein rings from the viewpoint of differential graded (=dg) enhancements. We show that an equivalence between the graded singularity category of a symmetric order and the derived category of an algebra automatically gives an equivalence between the ungraded singularity category and the cluster category. This is based on a structure theorem for certain Adams graded dg algebras with an enhanced Calabi-Yau property.

1. Introduction

The subject of this article is tilting and cluster tilting theory for Cohen-Macaulay representations of Gorenstein rings from the viewpoint of differential graded (dg) enhancements. Let R be a Gorenstein ring in a suitable sense; for example, a commutative Gorenstein ring or a finite dimensional Iwanaga-Gorenstein algebra over a field. One of the main objects of our interest is the *singularity category* of R defined as

$$\mathrm{D}_{\mathrm{sg}}(R) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, R)/\,\mathrm{per}\, R,$$

the Verdier quotient of the bounded derived category by the prefect derived category. It is a triangulated category, and is canonically equivalent to the stable category of Cohen-Macaulay modules. When R is graded we also consider the graded singularity category $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) = \mathrm{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} R)/\,\mathrm{per}^{\mathbb{Z}} R$.

On the other hand, some natural triangulated categories associated to an algebra A are the *derived category* $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ and the *cluster category* $\mathrm{C}_d(A)$. They have played an essential role in recent developments of cluster theory. While the derived category is a triangulated category of the canonical form endowed with a tilting object, the d-cluster category $\mathrm{C}_d(A)$ of A can be seen as a counterpart for d-Calabi-Yau triangulated categories with d-cluster tilting objects.

The (big) problem we tackle is the following (see [3]), which is to compare these kinds of triangulated categories.

Problem 1. Let R be a graded Gorenstein ring. Find a commutative diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) & \xrightarrow{\simeq} & \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) \\ \downarrow & & \downarrow \\ \mathrm{D}_{\mathrm{sg}}(R) & \xrightarrow{\simeq} & \mathrm{C}_d(A). \end{array}$$

The detailed version of this paper will be submitted for publication elsewhere.

Such equivalences realize the singularity categories as certain "canonical forms" of triangulated categories, and provide a mutual understanding of representation theories of R and A.

While the "graded–derived" equivalence naturally predicts the "ungraded–cluster" equivalence, it often involves technical difficulties to actually prove. The central ingredient of our discussion is then to use enhancements, which serves nicely to overcome these difficulties.

Throughout we fix a base field k which we assume to be perfect (for simplicity), and denote $D = \mathrm{Hom}_k(-, k)$.

2. Enhancements

Let us recall the definition of the cluster category of a finite dimensional algebra as an orbit category of the derived category [1, 4], which also tells us what the enhancement of it is.

Let $d \geq 1$ and let A be a finite dimensional algebra of finite global dimension. Replacing $DA[-d]$ by its bimodule projective resolution $X \to DA[-d]$, the autoequivalence $- \otimes_A^{\mathrm{L}} DA[-d] \colon \mathrm{D^b}(\mathrm{mod}\, A) \to \mathrm{D^b}(\mathrm{mod}\, A)$ lifts to the dg enhancement

$$- \otimes_A X \colon \mathcal{C}^b(\mathrm{proj}\, A) \to \mathcal{C}^b(\mathrm{proj}\, A).$$

Then the *dg orbit algebra* $\mathbf{\Gamma}_d(A)$ of A by $DA[-d]$ is defined by the colimit

$$\mathrm{colim}\left(\bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(X^n, A) \longrightarrow \bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(X^n, X) \longrightarrow \bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(X^n, X^2) \longrightarrow \cdots \right),$$

where X^n is the n-fold tensor product of X over A, and the transition morphisms are induced by $- \otimes_A X$.

Remark 2. (1) Our dg orbit algebra $\mathbf{\Gamma}_d(A)$ is nothing but the endomorphism algebra of A in the dg orbit category $\mathcal{C}^b(\mathrm{proj}\, A)/- \otimes_A X$ of Keller [4].

(2) The first term $\bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(X^n, A)$ is exactly the $(d+1)$-*derived preprojective algebra* (or the $(d+1)$-*Calabi-Yau completion*) $\mathbf{\Pi}_{d+1}(A)$ of A [5]. There is a morphism $\mathbf{\Pi}_{d+1}(A) \to \mathbf{\Gamma}_d(A)$ which is a localization of dg algebras.

Now we can define cluster categories in terms of enhancements.

Definition 3. The *d-cluster category* of A is the derived category $\mathrm{per}\, \mathbf{\Gamma}_d(A)$.

The following example is almost trivial, but useful enough to helps us understand what these enhancements look like.

Example 4. Let $d \geq 1$ be arbitrary and $A = k$. Then the autoequivalence $- \otimes_A^{\mathrm{L}} DA[-d] \colon \mathrm{D^b}(\mathrm{mod}\, A) \to \mathrm{D^b}(\mathrm{mod}\, A)$ is just $[-d]$, so it lifts to an *automorphism*

$$[-d] \colon \mathcal{C}^b(\mathrm{proj}\, k) \to \mathcal{C}^b(\mathrm{proj}\, k)$$

of dg categories. Therefore, it has an inverse and we can get rid of the colimit in the definition of dg orbit algebras. It gives $\mathbf{\Pi}_{d+1}(k) = k[x]$ with $\deg x = -d$ and $\mathbf{\Gamma}_d(k) = k[x, x^{-1}]$ with $\deg x = -d$, and the map $\mathbf{\Pi}_{d+1}(k) \to \mathbf{\Gamma}_d(k)$ is the obvious inclusion.

Note that the above definition of dg orbit algebras is valid for arbitrary equivalence given by bimodules, so let us give a generalization. Let A be a dg algebra and U an (A,A)-bimodule inducing an equivalence $-\otimes^{\mathrm{L}}_A U\colon \operatorname{per} A \to \operatorname{per} A$. Replace U by its bimodule cofibrant resolution $\widetilde{U} \to U$ and put

$$\mathbf{\Gamma}(A,U) := \operatorname{colim} \left(\bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(\widetilde{U}^n, A) \longrightarrow \bigoplus_{n\geq 0} \mathcal{H}\mathrm{om}_A(\widetilde{U}^n, \widetilde{U}) \longrightarrow \cdots \right),$$

which we call the *dg orbit algebra of A by U*. Let us give an important observation in this generality.

Proposition 5. (1) *The dg orbit algebra has a natural (additional, non-homological) grading given by*

$$\mathbf{\Gamma}(A,U)_i = \operatorname*{colim}_{m\gg 0} \left(\mathcal{H}\mathrm{om}_A(\widetilde{U}^{i+m}, \widetilde{U}^m) \longrightarrow \mathcal{H}\mathrm{om}_A(\widetilde{U}^{i+m+1}, \widetilde{U}^{m+1}) \longrightarrow \cdots \right).$$

(2) *The graded derived category* $\operatorname{per}^{\mathbb{Z}} \mathbf{\Gamma}(A,U)$ *is generated by a single object* $\mathbf{\Gamma}(A,U)$.
(3) *There exists a commutative diagram*

$$\begin{array}{ccc} \operatorname{per}^{\mathbb{Z}} \mathbf{\Gamma}(A,U) & \overset{\simeq}{\text{———}} & \operatorname{per} A \\ \downarrow & & \downarrow \\ \operatorname{per} \mathbf{\Gamma}(A,U) & \overset{\simeq}{\text{——}} & (\operatorname{per} A/-\otimes^{\mathrm{L}}_A U)_{\triangle}, \end{array}$$

where $(-)_{\triangle}$ means the canonical triangulated hull.

We will refer to the additional grading in (1) as *Adams degree.* Thus an Adams graded dg algebra is a bigraded algebra with a differential of degree $(1,0)$, where the first component is the (co)homological degree, and the second one is the Adams degree.

In particular, if A is a finite dimensional algebra and $U = DA[-d]$ we obtain the following commutative diagram involving cluster categories.

$$\begin{array}{ccc} \operatorname{per}^{\mathbb{Z}} \mathbf{\Gamma}_d(A) & \overset{\simeq}{\text{——}} & \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \\ \downarrow & & \downarrow \\ \operatorname{per} \mathbf{\Gamma}_d(A) & \overset{\simeq}{\text{———}} & \mathrm{C}_d(A) \end{array}$$

3. Results

We have observed that the projection functor $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{C}_d(A)$ from the derived category to the cluster category can be identified with the forgetful functor in terms of Adams graded dg algebras enhancing the cluster category. Our general main result is the following characterization of dg orbit algebras among Adams graded dg algebras. We say dg algebras A and B are *derived Morita equivalent* if there is an (A,B)-bimodule X inducing an equivalence $-\otimes^{\mathrm{L}}_A X\colon \mathrm{D}(A) \to \mathrm{D}(B)$. It means A and B can be reasonably idetified.

Theorem 6. *Let Γ be an Adams graded dg algebra. Suppose that* $\operatorname{per}^{\mathbb{Z}}\Gamma$ *is generated by a singule object M and put* $A = \operatorname{RHom}^{\mathbb{Z}}_{\Gamma}(M,M)$ *and* $U = \operatorname{RHom}^{\mathbb{Z}}_{\Gamma}(M, M(-1))$.

(1) *There dg algebras* Γ *and* $\mathbf{\Gamma}(A, U)$ *are derived Morita equivlaent.*

Suppose moreover that we have $D\Gamma \simeq \Gamma(-a)[d]$ *in* $\mathrm{D}^{\mathbb{Z}}(\Gamma^e)$ *for some* $a, d \in \mathbb{Z}$.

(2) *We have* $U^{\otimes^{\mathrm{L}}_A a} = DA[-d]$, *so that* $- \otimes^{\mathrm{L}}_A U$ *gives an* a*-th root of* $- \otimes^{\mathrm{L}}_A DA[-d]$ *on* $\operatorname{per} A$.

Let us apply the above result to Cohen-Macaulay representation theory. Let R be a commutative Gorenstein ring and Λ an R-algebra. We asy Λ is an *R-order* if Λ is finitely generated and (maximal) Cohen-Macaulay as an R-module. An R-order Λ is *symmetric* if $\mathrm{Hom}_R(\Lambda, R) \simeq \Lambda$ as bimodules over Λ.

Now we suppose that our Gorenstein ring R and symmetric R-order Λ is graded; $\Lambda = \bigoplus_{i \in \mathbb{Z}} \Lambda_i$ so that each Λ_i is finite dimensional. Assume furthermore that Λ has Gorenstein parameter a and has only an isolated singularity, that is, $\mathrm{gl.dim}\, \Lambda_{\mathfrak{p}} = \mathrm{ht}\, \mathfrak{p}$ for all prime ideals with $\mathrm{ht}\, \mathfrak{p} < \dim R =: d+1$. Then one obtains the following class of Adams graded dg algebras with the enhanced Calabi-Yau property.

Theorem 7. *Let* $\mathcal{C}$ *be the canonical enhancement of* $\mathrm{D}_{\mathrm{sg}}(\Lambda)$ *which is naturally an Adams graded dg category. Then we have*

$$D\mathcal{C} \simeq \mathcal{C}(-a)[d]$$

in $\mathrm{D}^{\mathbb{Z}}(\mathcal{C}^e)$, *that is, we have a natural quasi-isomorphism* $D\mathcal{C}(M, N) \simeq \mathcal{C}(N, M)(-a)[d]$ *for each* $M, N \in \mathcal{C}$.

Combining the above two results, we conclude that any "graded–derived" equivalence automatically gives a natural "ungraded–cluster" equivalence, reducing Problem 1 to finding the former equivalence.

Theorem 8. *Let* $R = \bigoplus_{i \in \mathbb{Z}} R_i$ *be a commutative Gorenstein graded ring of dimension* $d+1$ *such that each* R_i *is finite dimensional over* k, *and let* $\Lambda = \bigoplus_{i \in \mathbb{Z}} \Lambda_i$ *be a symmetric graded* R*-order which has Gorenstein parameter* a *and has an isolated singularity. Suppose that* $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(\Lambda)$ *has a tilting object* M *and put* $A = \underline{\mathrm{End}}^{\mathbb{Z}}_{\Lambda}(M)$. *Then there exists a commutative diagram of equivalences*

$$\begin{array}{ccc}
\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(\Lambda) & \overset{\simeq}{\text{———}} & \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \\
\downarrow & & \downarrow \\
\mathrm{D}^{\mathbb{Z}/a\mathbb{Z}}_{\mathrm{sg}}(\Lambda) & \overset{\simeq}{\text{———}} & \mathrm{C}_d(A) \\
\downarrow & & \downarrow \\
\mathrm{D}_{\mathrm{sg}}(\Lambda) & \overset{\simeq}{\text{———}} & \mathrm{C}^{(1/a)}_d(A).
\end{array}$$

Here, the category $\mathrm{C}^{(1/a)}_d(A)$ is the triangulated hull of the orbit category of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$ by a naturally defined a-th root of $- \otimes^{\mathrm{L}}_A DA[-d]$ (c.f. [2]). The above result shows such "cluster categories" arise naturally in Cohen-Macaulay representations.

References

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