KRULL–GABRIEL DIMENSION OF COHEN–MACAULAY MODULES OVER HYPERSURFACES OF TYPE (A_∞)

NAOYA HIRAMATSU

Abstract. We calculate the Krull–Gabriel dimension of the functor category of the (stable) category of maximal Cohen–Macaulay modules over hypersurfaces of type (A_∞).

1. Introduction

The notion of a Krull–Gabriel dimension has been considered under a functorial approach viewpoint of representation theory of finite dimensional algebras. It was introduced by Gabriel[4] and has been studied by many authors including Geigle[5], Schröer[12] and others.

Definition 1 (Krull Gabriel dimension)**.** Let $\mathcal{A}$ be a abelian category. Define $\mathcal{A}_{-1} = 0$. For each $n \geqq 1$, let $\mathcal{A}_n$ be the category of all objects which are finite length in $\mathcal{A}/\mathcal{A}_{n-1}$. We define KGdim $\mathcal{A} = min\{n \mid \mathcal{A} = \mathcal{A}_n\}$ if such a minimum exists, and KGdim $\mathcal{A} = \infty$ else.

Let R be a commutative Cohen–Macaulay local ring and $\mathcal{C}(R)$ the category of maximal Cohen–Macaulay R-modules. In this note we study the Krull–Gabriel dimension of $\underline{\mathrm{mod}}(\mathcal{C}(R))$; the full subcategory of $\mathrm{mod}(\mathcal{C}(R))$ consisting of all functors with $F(R) = 0$.

Theorem 2. *Let R be a complete Cohen–Macaulay local ring. Then R is of finite representation type if and only if* KGdim $\underline{\mathrm{mod}}(\mathcal{C}(R)) = 0$.

Let k be an algebraically closed uncountable field of characteristic not two. Next we investigate the case when R is a hypersurface of type (A_∞), that is, R is isomorphic to the ring $k[\![x_0, x_1, x_2, \ldots, x_n]\!]/(f)$, where $f = x_1^2 + x_2^2 + \cdots + x_n^2$. It is known that R is of countable representation type [3].

Theorem 3. *Let k be an algebraically closed uncountable field of characteristic not two. Let R be a hypersurface of type (A_∞). Then* KGdim $\underline{\mathrm{mod}}(\mathcal{C}(R)) = 2$.

The study of the Krull–Gabriel dimension of maximal Cohen–Macaulay modules over a one-dimensional hypersurface of type (A_∞) is given by Puninski[11]. His study investigates the Krull–Gabriel dimension of the definable category of maximal Cohen–Macaulay modules in $\mathrm{Mod}(R)$, so that it is different from ours.

The detailed version of this note will be submitted for publication elsewhere.
The author was partly supported by JSPS KAKENHI Grant Number 18K13399 and 21K03213.

2. Preliminaries

Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a Serre subcategory of $\mathcal{A}$. We say that a full subcategory $\mathcal{S}$ of $\mathcal{A}$ is a Serre subcategory if $\mathcal{S}$ is closed under taking subobjects, quoteients and extension. Note that a category of finite-length objects is a Serre subcategory. The quotient category $\mathcal{A}/\mathcal{S}$ is defined as follows: The object of $\mathcal{A}/\mathcal{S}$ are the objects of $\mathcal{A}$ and $\mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(X,Y) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$ with $X' \subset X$, $Y' \subset Y$ and X/X', $Y' \in \mathcal{S}$. Then $\mathcal{A}/\mathcal{S}$ is an abelian category.

To show a simpleness of an object in a quotient category, the following lemma is useful.

Lemma 4. [6, Lemma 1.1] *Let $\mathcal{A}$ be an abelian category and $\mathcal{S}$ a Serre subcategory. The object X of $\mathcal{A}$ becomes simple in $\mathcal{A}/\mathcal{S}$ if X is not an object of $\mathcal{S}$ and if for each subobject V of X either V or X/V belongs to $\mathcal{S}$.*

Let R be a commutative Noetherian ring with a finite Krull dimension. We denote by $\mathrm{mod}(R)$ a category of finitely generated R-modules with R-homomorphisms.

Proposition 5. *Let R be a commutative Noetherian ring with a finite Krull dimension. Then* $\mathrm{KGdim}\ \mathrm{mod}(R) = \dim R$.

Proof. One can show that $R/\mathfrak{p}$ is a simple object in $\mathrm{mod}(R)/\mathrm{mod}(R)_{i-1}$ for a prime ideal $\mathfrak{p}$ with $i = \dim R/\mathfrak{p}$. Let M be a finitely generated R-module. Now we have a filtration of M: $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that $M_k/M_{k-1} \cong R/\mathfrak{p}_k$ with prime ideals $\mathfrak{p}_k$. This implies that $\mathrm{KGdim}\ M \leq min\{\dim R/\mathfrak{p}_k | k = 1, \cdots n\}$. Hence $\mathrm{KGdim}\ \mathrm{mod}(R) \leq inf\{\dim R/\mathfrak{p} | \mathfrak{p} \in \mathrm{Spec} R\} \leq \dim R$. On the other hand, take a minimal associated prime ideal $\mathfrak{p}$ of R, then $\dim R/\mathfrak{p} = \dim R$, so that $\dim R \leq \mathrm{KGdim}\ \mathrm{mod}(R)$. Therefore we obtain $\mathrm{KGdim}\ \mathrm{mod}(R) = \dim R$. □

Now we focus on a category of maximal Cohen-Macaulay (abbr.MCM) modules. In the rest of the note we always assume that $(R, \mathfrak{m})$ is a complete CM local ring. We denote by $\mathcal{C}(R)$ the full subcategory of $\mathrm{mod}(R)$ consisting of all MCM R-modules and by $\mathcal{C}_0(R)$ the full subcategory of $\mathcal{C}(R)$ consisting of all modules that are locally free on the punctured spectrum of R. We denote by $\underline{\mathcal{C}}(R)$ the stable category of $\mathcal{C}(R)$. The objects of $\underline{\mathcal{C}}(R)$ are the same as those of $\mathcal{C}(R)$, and the morphisms of $\underline{\mathcal{C}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M,N) = \mathrm{Hom}_R(M,N)/P(M,N)$ for $M, N \in \underline{\mathcal{C}}(R)$, where $P(M,N)$ denote the set of morphisms from M to N factoring through free R-modules. Since R is complete, $\mathcal{C}(R)$, thus $\underline{\mathcal{C}}(R)$, is a Krull-Schmidt category. For a finitely generated R-module M, we denote by $\mathrm{syz}^1_R(M)$ the reduced first syzygy of M.

Let us recall the full subcategory of the functor category of $\mathcal{C}(R)$ which is called the Auslander category. The Auslander category $\mathrm{mod}(\mathcal{C}(R))$ is the category whose objects are finitely presented contravariant additive functors from $\mathcal{C}(R)$ to a category of abelian groups and whose morphisms are natural transformations between functors. We denote by $\underline{\mathrm{mod}}(\mathcal{C}(R))$ the full subcategory $\mathrm{mod}(\mathcal{C}(R))$ consisting of functors F with $F(R) = 0$. Note that every object $F \in \underline{\mathrm{mod}}(\mathcal{C}(R))$ is obtained from a short exact sequence in $\mathcal{C}(R)$. Namely we have the short exact sequence $0 \to N \to M \to L \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ , N) \to \mathrm{Hom}_R(\ , M) \to \mathrm{Hom}_R(\ , L) \to F \to 0$$

is exact in $\mathrm{mod}(\mathcal{C}(R))$.

Let $0 \to Z \to Y \to X \to 0$ be an AR sequence in $\mathcal{C}(R)$. (For a theory of Auslander-Reiten (abbr. AR) sequences, we refer to [13].) Then the functor S_X defined by an exact sequence

$$0 \to \operatorname{Hom}_R(\ ,Z) \to \operatorname{Hom}_R(\ ,Y) \to \operatorname{Hom}_R(\ ,X) \to S_X \to 0$$

is a simple object in $\operatorname{mod}(\mathcal{C}(R))$ and all the simple objects in $\operatorname{mod}(\mathcal{C}(R))$ are obtained in this way ([13, Lemma 4.12]).

Let us show the first result of the note, which is an analogical result due to Auslander[2]. We say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable MCM R-modules. For a functor $F \in \operatorname{Mod}(\mathcal{C}(R))$, we denote by $\operatorname{Supp}(F)$ a set of isomorphism classes of indecomposable MCM modules M with $F(M) \neq 0$.

Theorem 6. *Let R be a complete CM local ring. Then R is of finite representation type if and only if* $\operatorname{KGdim} \underline{\operatorname{mod}}(\mathcal{C}(R)) = 0$.

Proof. Suppose that R is of finite representation type. According to [13, Chapter 13], every functor $F \in \underline{\operatorname{mod}}(\mathcal{C}(R))$ has finite length. Hence $\operatorname{KGdim} \underline{\operatorname{mod}}(\mathcal{C}(R)) = 0$. Conversely suppose that $\operatorname{KGdim} \underline{\operatorname{mod}}(\mathcal{C}(R)) = 0$. By [8, Lemma 2.1], there exists $X \in \mathcal{C}(R)$ such that $\underline{\operatorname{Hom}}_R(M,X) \neq 0$ for all non free MCM R-modules M. That is, $\operatorname{Supp}(\underline{\operatorname{Hom}}_R(-,X)) \cup \{R\} = \operatorname{Ind}(\mathcal{C}(R))$. Since $\underline{\operatorname{Hom}}_R(-,X) \in \underline{\operatorname{mod}}(\mathcal{C}(R))$, $\ell(\underline{\operatorname{Hom}}_R(-,X)) < \infty$ in $\underline{\operatorname{mod}}(\mathcal{C}(R))$. This implies that $|\operatorname{Supp}(\underline{\operatorname{Hom}}_R(-,X))| < \infty$. Hence R is of finite representation type. $\square$

Remark 7. We note that $\operatorname{KGdim} \operatorname{mod}(\mathcal{C}(R))$ is not always 0 if R is of finite representation type. Actually let $R = k[\![x]\!]$. Then $\mathcal{C}(R) = \operatorname{add}\{R\}$. Thus R is of finite representation type. Since $\operatorname{mod}(\mathcal{C}(R)) = \operatorname{mod}(R)$, we have the equality $\operatorname{KGdim} \operatorname{mod}(R) = \dim R = 1$ by Proposition 5.

3. Krull Gabriel dimension of $\underline{\operatorname{mod}}(k[\![x,y]\!]/(x^2))$

Let k be an algebraically closed uncountable field of characteristic not 2 and R a one-dimensional hypersurface of type (A_∞), that is, $R = k[\![x,y]\!]/(x^2)$. This section is devoted to calculate the Krull-Gabriel dimension of $\underline{\operatorname{mod}}(\mathcal{C}(R))$. It is known that R is of *countable* representation type, namely there exist infinitely but only countably many isomorphism classes of indecomposable MCM R-modules. The non free indecomposable MCM R-modules are as follows:

$$I_n = \operatorname{Coker}\begin{pmatrix} x & y^n \\ 0 & x \end{pmatrix} : R^{\oplus 2} \to R^{\oplus 2} \qquad I = \operatorname{Coker}(x) : R \to R.$$

See [3, Proposition 4.1]. First we state the main result in this section.

Theorem 8. *Let k be an algebraically closed uncountable field of characteristic not 2 and $R = k[\![x,y]\!]/(x^2)$. Then,* $\operatorname{KGdim} \underline{\operatorname{mod}}(\mathcal{C}(R)) = 2$.

To prove the theorem, we shall do some preparations.

Lemma 9. *Let R, I, I_n be as above. The following statements hold.*

(1) $\dim_k \underline{\operatorname{Hom}}_R(I_m, I_n) = \begin{cases} 2n & m \geq n, \\ 2m & m \leq n. \end{cases}$

(2) $\dim_k \underline{\operatorname{Hom}}_R(I, I_n) = \dim_k \underline{\operatorname{Hom}}_R(I_n, I) = n$ *for* $1 \leq n < \infty$.

(3) $\dim_k \underline{\operatorname{Hom}}_R(I, I) = \infty$. $\square$

One has an exact sequence

$$0 \to I_1 \xrightarrow{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}} I \oplus R \xrightarrow{(y\ \ -x)} I \to 0.$$

We consider the functor induced by the sequence;

$$(3.1) \qquad 0 \to \mathrm{Hom}_R(-, I_1) \to \mathrm{Hom}_R(-, I) \oplus \mathrm{Hom}_R(-, R) \to \mathrm{Hom}_R(-, I) \to H_1 \to 0.$$

We shall show the functor H_1 is a simple functor in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$.

Proposition 10. *The functor H_1 is simple in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$.*

Proof. By [14, Proposition 3.3], the exact sequence (3.1) induces the long exact sequence:

$$(3.2) \qquad \begin{array}{ccccccc} \longrightarrow & H_1 & \longrightarrow & 0 & & & \\ \longrightarrow & \underline{\mathrm{Hom}}_R(-, I_1) & \xrightarrow{\frac{x}{y}} & \underline{\mathrm{Hom}}_R(-, I) & \xrightarrow{y} & \underline{\mathrm{Hom}}_R(-, I) & \\ \longrightarrow & \underline{\mathrm{Hom}}_R(-, I_1[-1]) & \longrightarrow & \underline{\mathrm{Hom}}_R(-, I[-1]) & \longrightarrow & \underline{\mathrm{Hom}}_R(-, I[-1]). & \end{array}$$

For each indecomposable $X \in \mathcal{C}_0(R)$, since $\dim_k \underline{\mathrm{Hom}}_R(X, I_1)$ and $\dim_k \underline{\mathrm{Hom}}_R(X, I)$ is finite, we have $\dim_k H_1(X) = \frac{1}{2} \dim_k \underline{\mathrm{Hom}}_R(X, I_1) = 1$. Notice here again that $M \cong M[-1]$ for every MCM R-module M. Since $\underline{\mathrm{Hom}}_R(I, I) \cong k[\![y]\!]$, one has $H_1(I) \cong k[\![y]\!]/yk[\![y]\!]$. Consequently, we have $\dim_k H_1(X) = 1$ for all indecomposable $X \in \mathcal{C}(R)$.

Let $0 \to V \to H_1 \to C \to 0$ be an admissible exact sequence in $\underline{\mathrm{mod}}(\mathcal{C}(R))$. Since $V \in \underline{\mathrm{mod}}(\mathcal{C}(R))$, we have the exact sequence: $0 \to \mathrm{Hom}_R(-, Z) \to \mathrm{Hom}_R(-, Y) \to \mathrm{Hom}_R(-, X) \to V \to 0$. Then, for all $M \in \mathcal{C}_0(R)$,

$$\dim_k V(M) = \frac{1}{2} \left\{ \dim_k \underline{\mathrm{Hom}}_R(M, X) + \dim_k \underline{\mathrm{Hom}}_R(M, Z) - \dim_k \underline{\mathrm{Hom}}_R(M, Y) \right\}.$$

$X = I^{\oplus a_0} \oplus I_{l_1}^{\oplus a_1} \oplus \cdots \oplus I_{l_{l'}}^{\oplus a_{l'}}$, $Y = I^{\oplus b_0} \oplus I_{m_1}^{\oplus b_1} \oplus \cdots \oplus I_{m_{m'}}^{\oplus b_{m'}}$ and $Z = I^{\oplus c_0} \oplus I_{n_1}^{\oplus c_1} \oplus \cdots \oplus I_{n_{n'}}^{\oplus c_{n'}}$. We put $m = max\{l_1, ..., l_{l'}, m_1, ..., m_{m'}, n_1, ..., n_{n'}\}$. For $m \leqq n < \infty$,

$$\dim_k V(I_n) = \frac{1}{2} \left(\sum_i^{l'} m \cdot a_i + \sum_i^{n'} m \cdot c_i - \sum_i^{m'} m \cdot b_i \right).$$

This equation yields that $\dim_k V(I_n)$ are 0 or 1 for $m \leqq n < \infty$ since V is a subfunctor of H_1. Assume that $\dim_k V(I_n) = 0$ for $m \leqq n$. Then $V(I_n) = 0$ except for a finite number of I_n. Namely $\mathrm{Supp}(V)$ is a finite set, and we shall show $I \notin \mathrm{Supp}(V)$. If it does, V is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_0$. Assume that $I \in \mathrm{Supp}(V)$. For $I' \in \mathrm{Supp}(V) \bigcap \mathcal{C}_0(R)$, there is an epimorphism from $V \to S_{I'}$. (See the proof of [13, (4.12)].) Put the kernel of the epimorphism as V'. Then $V' \in \underline{\mathrm{mod}}(\mathcal{C}(R))$ and $\mathrm{Supp}(V') = \mathrm{Supp}(V) \backslash \{I'\}$. Repeating the procedure, we obtain the functor $\tilde{V} \in \underline{\mathrm{mod}}(\mathcal{C}(R))$ such that $\mathrm{Supp}(\tilde{V}) = \{I\}$ and $\dim_k \tilde{V}(I) = 1$. It yields that $\tilde{V}$ is a simple functor with $\tilde{V}(I) \neq 0$, so that the AR sequence ending in I exists ([13, (4.13)]). Namely $I \in \mathcal{C}_0(R)$ ([13, (3.4)]). This is a contradiction. Hence $I \notin \mathrm{Supp}(V)$.

Assume that $\dim_k V(I_n) = 1$ for $m \leqq n$. Then $\dim_k C(I_n) = 0$ for $m \leqq n$. Apply the same argument for C and we also conclude that C is contained in $\underline{\mathrm{mod}}(\mathcal{C}(R))_0$. Consequently we get the assertion. ∎

Remark 11. Since H_1 is a subfunctor of $\underline{\mathrm{Hom}}_R(-, I_1)$, we have an exact sequence in $\underline{\mathrm{mod}}(\mathcal{C}(R))$:

$$0 \to H_1 \to \underline{\mathrm{Hom}}_R(-, I_1) \to H_1' \to 0.$$

By virtue of Lemma 9 and a calculation in the proof of Proposition 10, $\dim_k H_1'(I_n) = 1$ for all n and $\dim_k H_1'(I) = 0$. By using the same argument of Proposition 10, one can also show that H_1' is a simple functor in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$. Therefore, $\ell(\underline{\mathrm{Hom}}_R(-, I_1)) = 2$ in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$.

Remark 12. In the Grothendieck group of $\underline{\mathrm{mod}}(\mathcal{C}(R))$, an AR sequence gives the equality $[\underline{\mathrm{Hom}}_R(-, I_{n+1})] + [\underline{\mathrm{Hom}}_R(-, I_{n-1})] = 2[\underline{\mathrm{Hom}}_R(-, I_n)] - 2[S_{I_n}]$. Combing the equation with Remark 11, one can show that $\ell(\underline{\mathrm{Hom}}_R(-, I_n)) = 2n$ in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$ for $n \geq 1$. By [1, Proposition 2.1 (1)], there is an exact sequence $0 \to I \to I_n \to I \to 0$ for $n \geq 1$. Then $2\ell(\underline{\mathrm{Hom}}_R(-, I)) \geq \ell(\underline{\mathrm{Hom}}_R(-, I_n))$ in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_0$. This yields that $\underline{\mathrm{Hom}}_R(-, I) \notin \underline{\mathrm{mod}}(\mathcal{C}(R))_1$.

Proposition 13. *The functor* $\underline{\mathrm{Hom}}_R(-, I)$ *is simple in* $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_1$.

Proof. Let $0 \to V \to \underline{\mathrm{Hom}}_R(-, I) \to C \to 0$ be an admissible sequence in $\underline{\mathrm{mod}}(\mathcal{C}(R))$. Since $V \in \underline{\mathrm{mod}}(\mathcal{C}(R))$, there is an exact sequence $0 \to \mathrm{Hom}_R(-, Z) \to \mathrm{Hom}_R(-, Y) \to \mathrm{Hom}_R(-, X) \to V \to 0$ for some $X, Y, Z \in \mathcal{C}(R)$. If $X \in \mathcal{C}(R)_0$, $V \in \underline{\mathrm{mod}}(\mathcal{C}(R))_1$ because V is an image of $\underline{\mathrm{Hom}}_R(-, X)$ ([13, (4.16)]). Thus the claim holds. Assume that X contains I as a direct summand. After the several observations, we may assume that C has the presentation: $\underline{\mathrm{Hom}}_R(-, I^{\oplus l}) \to \underline{\mathrm{Hom}}_R(-, I) \to C \to 0$. By investigating the presentation minutely, one can also show that C has the resolution: $\underline{\mathrm{Hom}}_R(-, I_n) \to \underline{\mathrm{Hom}}_R(-, I) \to \underline{\mathrm{Hom}}_R(-, I) \to C \to 0$. This implies that C is a subfunctor of $\underline{\mathrm{Hom}}_R(-, I_n[1]) \cong \underline{\mathrm{Hom}}_R(-, I_n)$. Consequently, $\underline{\mathrm{Hom}}_R(-, I)$ is simple in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_1$. □

Proof of Theorem 8. For each $F \in \underline{\mathrm{mod}}(\mathcal{C}(R))$, we have a epimorphism $\mathrm{Hom}_R(-, X) \to F \to 0$. In particular, the epimorphism

$$\underline{\mathrm{Hom}}_R(-, X) \to F \to 0$$

exists, where $X \in \mathcal{C}(R)$. From the former propositions, $\underline{\mathrm{Hom}}_R(-, X)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_2$ and so is F. It induces that KGdim $\underline{\mathrm{mod}}(\mathcal{C}(R)) = 2$. □

4. Knörrer's periodicity

In this section we investigate how a Krull-Gabriel dimension changes with Knörrer's periodicity. We recall some observations given in [10, 9].

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories with a functor $\mathcal{A} : \mathcal{C} \to \mathcal{D}$. Then $\mathcal{A}$ induces the functor $\mathcal{A} : \underline{\mathrm{mod}}(\mathcal{C}) \to \underline{\mathrm{mod}}(\mathcal{D})$ by $\mathcal{A}(\mathrm{Hom}_{\mathcal{C}}(-, C)) = \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{A}(C))$. That is, for $F \in \underline{\mathrm{mod}}\mathcal{C}$ with $0 \to \mathrm{Hom}_{\mathcal{C}}(-, Z) \to \mathrm{Hom}_{\mathcal{C}}(-, Y) \to \mathrm{Hom}_{\mathcal{C}}(-, X) \to F \to 0$, $\mathcal{A}(F)$ is defined by $0 \to \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{A}(Z)) \to \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{A}(Y)) \to \mathrm{Hom}_{\mathcal{D}}(-, \mathcal{A}(X)) \to \mathcal{A}(F) \to 0$.

Lemma 14. *Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories with functors $\mathcal{A} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{B} : \mathcal{D} \to \mathcal{C}$. Suppose that $(\mathcal{B}, \mathcal{A})$ is an adjoint pair of functors. Then the induced functor $\mathcal{A} : \underline{\mathrm{mod}}(\mathcal{C}) \to \underline{\mathrm{mod}}(\mathcal{D})$ is an exact functor.*

Proof. By the adjointness of $(\mathcal{B}, \mathcal{A})$, one can show that $\mathcal{A}(F)(-) \cong F(\mathcal{B}(-))$ for $F \in \underline{\mathrm{mod}}(\mathcal{C})$. The assertion follows from the isomorphism. $\square$

Let R be a hypersurface, that is, $R = S/(f)$ where $S = k[\![x_0, x_1, \cdots, x_n]\!]$ is a formal power series ring with a maximal ideal $\mathfrak{m}_S = (x_0, x_1, \cdots, x_n)$ and $f \in \mathfrak{m}_S$. For the ring R, we denote $R^\sharp = S[\![z]\!]/(f + z^2)$. Then the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on $R^\sharp$ by $\sigma : z \to -z$. Denote the skew group ring by $R^\sharp * G$. We also denote by $\mathcal{C}(R)$, $\mathcal{C}(R^\sharp)$, $\mathcal{C}(R^\sharp * G)$ the category of MCM R-, $R^\sharp$- and $R^\sharp * G$-modules respectively. For M in $\mathcal{C}(R^\sharp)$ and the involution σ in G, we define an $R^\sharp$-module $\sigma^* M$ by $M = \sigma^* M$ as a set and $r \circ m = \sigma(r)m$. For the detail, we refer to [9, Section 2].

Theorem 15. [9, Proposition 2.1, Remark 2.2, Proposition 2.4, Lemma 2.5] *Let R, $R^\sharp * G$, $R^\sharp$ be as above. We have the functors:*

$$\mathcal{C}(R) \xrightarrow{\Omega} \mathcal{C}(R^\sharp * G) \underset{\mathcal{F}}{\overset{ad}{\rightleftarrows}} \mathcal{C}(R^\sharp),$$

where the functor $\Omega(-)$ is defined by $\mathrm{syz}^1_{R^\sharp}(-)$, $\mathcal{F}$ is a forget-functor and $ad(-) = - \otimes_{R^\sharp} R^\sharp * G$ is its adjoint. Then, for $X \in \mathcal{C}(R)$ and $Y \in \mathcal{C}(R^\sharp)$, the following statements hold.

(1) *The functor Ω gives the categorical equivalence.*
(2) *$\Omega^{-1} \circ ad \circ F \circ \Omega$ is equivalent to the functor $X \mapsto X \oplus \mathrm{syz}^1_R(X)$.*
(3) *$F \circ \Omega \circ \Omega^{-1} \circ ad$ is equivalent to the functor $Y \mapsto Y \oplus \sigma * Y$.* $\square$

Lemma 16. [10, Theorem 3.2] *Let Ω, $\mathcal{F}$ and ad be as above. Set $\mathcal{A} = \mathcal{F} \circ \Omega$ and $\mathcal{B} = \Omega^{-1} \circ ad$. Then $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ are adjoint pairs.*

Proposition 17. *Let $R = S/(f)$ be a hypersurface and $\mathcal{A}$, $\mathcal{B}$ as in Lemma 16. Suppose that $\mathcal{A}(F) \in \underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_{n-1}$ for each $F \in \underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Then $\mathcal{A}(F)$ is contained in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_n$ for a simple functor $S \in \underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$.*

Proof. Let S be a simple functor in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Assume that $\mathcal{A}(S)$ is not simple. Then we have an exact sequence of functors $0 \to V \to \mathcal{A}(S) \to S' \to 0$ such that S' is simple in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))/\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_{n-1}$. Apply $\mathcal{B}$ to the sequence, one has $0 \to \mathcal{B}(V) \to \mathcal{B} \circ \mathcal{A}(S) \to \mathcal{B}(S') \to 0$. Since $\mathcal{B} \circ \mathcal{A}(S) \cong S \oplus S[-1]$ (notice that the functor $S[-1]$ is also a simple functor), one can show that $\mathcal{B}(V)$ and $\mathcal{B}(S')$ are simple in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Now we shall show V is also simple in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))/\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_{n-1}$. Let $0 \to V' \to V \to C \to 0$ be an admissible sequence in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))$. Then we obtain the exact sequence in $\underline{\mathrm{mod}}(\mathcal{C}(R))$: $0 \to \mathcal{B}(V') \to \mathcal{B}(V) \to \mathcal{B}(C) \to 0$. Since $\mathcal{B}(V)$ is simple $\mathcal{B}(V')$ or $\mathcal{B}(C)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Assume that $\mathcal{B}(V')$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Notice that $\mathcal{A} \circ \mathcal{B}(V') \cong V' \oplus \sigma * V'$, so that V' is a direct summand of $\mathcal{A} \circ \mathcal{B}(V')$. By the assumption, $\mathcal{A} \circ \mathcal{B}(V')$ is finite length, and so is V'. Therefore V' is a simple functor in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))/\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_{n-1}$. The same arguments are valid for the case that $\mathcal{B}(C)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. $\square$

Proposition 18. *Suppose that $\mathcal{A}(S)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_n$ for each simple functor S in $\underline{\mathrm{mod}}(\mathcal{C}(R))/\underline{\mathrm{mod}}(\mathcal{C}(R))_{n-1}$. Then $\mathcal{A}(F)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_n$ for each F in $\underline{\mathrm{mod}}(\mathcal{C}(R))_n$.*

Proof. Apply $\mathcal{A}$ to the filtration of F. $\square$

Remark 19. As mentioned in [9, Corollary 2.10], a simple functor S in $\underline{\mathrm{mod}}(\mathcal{C}(R))$ goes to a length-finite functor in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))$. Namely $\mathcal{A}(S) \in \underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_0$. Conversely a simple functor S' in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))$ also goes to a functor in $\underline{\mathrm{mod}}(\mathcal{C}(R))_0$.

Finally we achieve the main theorem of this note.

Corollary 20. *Let k be an algebraically closed uncountable field of characteristic not two. Let R be a hypersurface of type (A_∞) . Then* $\mathrm{KGdim}\ \underline{\mathrm{mod}}(\mathcal{C}(R)) = 2$.

Proof. Let $R = k[\![x, y]\!]/(x^2)$. Summing up Proposition 17, 18 and Remark 19, one can see that $\mathcal{A}$ gives a functor $\underline{\mathrm{mod}}(\mathcal{C}(R))_n \to \underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_n$. For each $F \in \underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))$, $\mathcal{B}(F)$ is in $\underline{\mathrm{mod}}(\mathcal{C}(R))_2$. Thus $\mathcal{A} \circ \mathcal{B}(F) = F \oplus \sigma^* F$ in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_2$, so that F is in $\underline{\mathrm{mod}}(\mathcal{C}(R^\sharp))_2$. This observation yields that the assertion holds for the hypersurfaces of all dimensions. ☐

Remark 21. In [7], we also calculate the Krull–Gabriel dimension of the functor category of the category of MCM modules over hypersurfaces of type (D_∞).

References

[1] Araya, T, Iima, K.-i. and Takahashi, T., *On the structure of Cohen–Macaulay modules over hypersurfaces of countable Cohen–Macaulay representation type*, J. Algebra. **361** (2012), 213–224.

[2] Auslander, M, *A functorial approach to representation theory*, Representations of algebras (ed. M. Auslander, E. Luis), Springer Lecture Notes in Math. **944** (1980), 105–179.

[3] Buchweitz, R.-O., Greuel, G.-M. and Schreyer, F.-O., *Cohen-Macaulay modules on hypersurface singularities. II.* Invent. Math. **88**, no. 1, 165–182 (1987).

[4] Gabriel, P., *Des catégories abéliennes*, Bull. Soc. math. France **90** (1962), 323–448.

[5] Geigle, W., *The Krull-Gabriel dimension of the representation theory of a tame hereditary Artin algebra and applications to the structure of exact sequences*, Manuscripta Math **54**, 83–106 (1985).

[6] Gordon, R. and Robson, J.C., *The Gabriel dimension of a module*, J. Algebra, Volume **29**, Issue 3, (1974) 459–473.

[7] Hiramatsu, N., *Krull Gabriel dimension of Cohen–Macaulay modules over hypersurfaces of countable Cohen-Macaulay representation type*, in preparation.

[8] Hiramatsu, N., *Relations for Grothendieck groups of Gorenstein rings*, Proc. Amer. Math. Soc. **145**, (2017) 559–562.

[9] Knörrer, H, *Cohen-Macaulay modules on hypersurface singularities I.* Invent Math **88**, (1987) 153–164.

[10] Reiten, I. and Riedtmann, C, *Skew group algebras in the representation theory of Artin algebras.* J. Algebra **92**, (1985) 224–282.

[11] Puninski, G., *The Ziegler Spectrum and Ringel's Quilt of the A-infinity Plane Curve Singularity*, Algebr Represent Theor **21**, 419–446 (2018).

[12] Schröer, J., *On the Krull-Gabriel dimension of an algebra.* Math Z **233**, 287-303 (2000).

[13] Yoshino, Y., *Cohen–Macaulay Modules over Cohen–Macaulay Rings, London Mathematical Society*, Lecture Note Series, **146**, *Cambridge University Press, Cambridge*, 1990.

[14] Yoshino, Y., *A functorial approach to modules of G-dimension zero*, Illinois J. Math. **49** (2) 345–367, Summer 2005.

General Education Program. National Institute of Technology, Kure College
2-2-11, Agaminami, Kure Hiroshima, 737-8506 Japan

Email address: `hiramatsu@kure-nct.ac.jp`