ALGEBRAIC STABILITY THEOREM FOR DERIVED CATEGORIES OF ZIGZAG PERSISTENCE MODULES

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Abstract. We study distances on zigzag persistence modules from the viewpoint of derived categories and Auslander–Reiten quivers. It is known that the derived category of persistence modules is derived equivalent to that of arbitrary zigzag persistence modules. Through this derived equivalence, we define and compute distances on the derived category of arbitrary zigzag persistence modules and prove an algebraic stability theorem. We also compare our distance with the other distances.

1. Introduction

This article is based on the paper `arXiv:2006.06924` [15], which is an interaction between representation theory of algebras and topological data analysis, particularly the robustness for noises of data.

Topological data analysis has recently become popular for studying the shape of data in various research areas (see [16] for example; also see `https://www.jst.go.jp/pr/announce/20160614/index.html`). Persistent homology [11] is one of the leading tools in topological data analysis. It provides a multi-scale analysis of the topological features of a given data set with the so-called persistence diagram as its output. Unlike ordinary homology, it is significant that a stability theorem holds for persistent homology [10].

The algebraic structure of persistent homology is expressed by the notion of *persistence modules*, which are representations of an equioriented A_n-type quiver [7]. With this notion, the stability theorem is generalized in a completely algebraic manner. It is called an algebraic stability theorem (AST; see [8], [1]). Namely, the AST guarantees that the persistence diagram is robust to changes in the given persistence module.

Moreover, a *zigzag persistence module* [7], a representation of an A_n-type quiver with arbitrary orientation, can be applied to address various situations which are not covered by the theory of ordinary persistence module (e.g. time-series data). Our motivation is to derive an AST for zigzag persistence modules. Botnan and Lesnick proved such a theorem by embedding the category of (purely) zigzag persistence modules into that of 2D block decomposable persistence modules [6]. Note that the zigzag persistence modules in [6] are *purely zigzag ones*, representations of a purely zigzag A_n-type quiver, in our convention.

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Here, we adopt a different approach: we study distances on zigzag persistence modules from the viewpoint of derived categories and Auslander–Reiten quivers.

For two persistence modules M, N, we can define the *interleaving distance* d_I between M and N. We denote by $\mathcal{B}(M)$ the *persistence diagram* of M, which consists of the indecomposable representations (we call them *intervals*) in the indecomposable decomposition of M. Then, the interleaving distance induces the *bottleneck distance* d_B between $\mathcal{B}(M)$ and $\mathcal{B}(N)$. Comparing these distances, the following holds.

Theorem 1 (AST [8],[1], see Theorem 8)**.** $d_B(\mathcal{B}(M), \mathcal{B}(N)) \leq d_I(M, N)$.

The distances d_I, d_B can be extended to the derived setting d_I^D, d_B^D. Thus, we obtain the following theorem for cochain complexes $X^\bullet, Y^\bullet$ of the derived category of persistence modules.

Theorem 2 (Derived AST [15], see Theorem 22)**.** $d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) \leq d_I^D(X^\bullet, Y^\bullet)$.

The derived category of persistence modules is derived equivalent to that of zigzag persistence modules, depending on a classical tilting module. Through this derived equivalence, we define and compute distances on the derived category of zigzag persistence modules and prove an algebraic stability theorem.

Proposition 3 ([15], see Proposition 27)**.** *For the derived category of zigzag persistence modules, an AST holds.*

As a consequence, an AST holds for zigzag persistence modules since the category of zigzag persistence modules is embedded in the derived category as a full subcategory.

Finally, we also compare our distance with the distance for purely zigzag persistence modules introduced by Botnan–Lesnick [6] and the sheaf-theoretic convolution distance due to Kashiwara–Schapira [18].

2. Preliminaries

Throughout this article, $\Bbbk$ denotes an algebraically closed field, and all vector spaces, algebras, and linear maps are assumed to be finite-dimensional $\Bbbk$-vector spaces, finite-dimensional $\Bbbk$-algebras, and $\Bbbk$-linear maps, respectively. Furthermore, all categories and functors are assumed to be additive. In addition, a distance on a set X means an extended pseudometric. Specifically, it is a function $d\colon X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for every $x, y, z \in X$,

(1) $d(x, x) = 0$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, z) \leq d(x, y) + d(y, z)$ if $d(x, y), d(y, z) < \infty$.

A *quiver* Q is a directed graph. Formally, a quiver Q is a quadruple $Q = (Q_0, Q_1, s, t)$ of sets Q_0 of vertices and Q_1 of arrows, and maps $s, t\colon Q_1 \to Q_0$. A quiver Q is *finite* if Q_0 and Q_1 are finite.

Here, we introduce the A_n-type quiver $A_n(a)$ with orientation a, whose underlying graph is the Dynkin diagram of type $A : 1 — 2 — \cdots — n$ for $n \in \mathbb{N}$. Then $A_n(a)$ is the quiver

$$1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow n, \tag{2.1}$$

where $\leftrightarrow$ means $\to$ or $\leftarrow$ assigned by the orientation a. In this article, the following A_n-type quivers with certain orientations are frequently used. The A_n-type quiver with equi-orientation

$$1 \to 2 \to \cdots \to n \tag{2.2}$$

is called the *equioriented A_n-type quiver*, which is denoted by $A_n(= A_n(e))$. The A_n-type quiver with alternating orientation is called a *purely zigzag A_n-type quiver*, which is denoted by $A_n(z)$. Moreover, if the vertex 1 of a purely zigzag A_n-type quiver Q is a sink vertex, Q is denoted by $A_n(z_1)$. Otherwise, it is denoted by $A_n(z_2)$. Namely, $A_n(z_1)$ is the following quiver:

$$1 \leftarrow 2 \to 3 \leftarrow \cdots \to n \text{ if } n \text{ is odd, } 1 \leftarrow 2 \to 3 \leftarrow \cdots \leftarrow n \text{ if } n \text{ is even,} \tag{2.3}$$

and $A_n(z_2)$ is the following quiver:

$$1 \to 2 \leftarrow 3 \to \cdots \leftarrow n \text{ if } n \text{ is odd, } 1 \to 2 \leftarrow 3 \to \cdots \to n \text{ if } n \text{ is even.} \tag{2.4}$$

A *representation* M of a quiver Q is a family of vector spaces M_x at each vertex $x \in Q_0$ and linear maps M_α on each arrow $\alpha \in Q_1$.

For a representation M of Q, the *dimension* of M is defined by $\dim M := \sum_{x \in Q_0} \dim M_x$. All representations M are assumed to be *pointwise finite-dimensional*, namely $\dim M_x < \infty$ for each $x \in Q_0$. When Q is finite, this is just *finite-dimensional*, that is, $\dim M < \infty$.

The abelian category of representations of Q is denoted by $\operatorname{rep}_{\Bbbk} Q$. Note that $\operatorname{rep}_{\Bbbk} Q$ is a Krull-Schmidt category when Q is finite. More generally, when Q is the infinite zigzag quiver (see Section 5.1), any $M \in \operatorname{rep}_{\Bbbk} Q$ has unique indecomposable (infinite) decomposition up to permutations and isomorphisms by Krull-Schmidt-Remak-Azumaya Theorem since every indecomposable representation of Q is an interval one (see [5]).

A poset can be identified with a quiver with relations. Then, for a poset, we use the same notation as quivers.

For an abelian category $\mathcal{A}$, $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ denotes the bounded derived category of $\mathcal{A}$ and $\Gamma(\mathcal{A})$ (resp. $\Gamma(\mathsf{D}^{\mathrm{b}}(\mathcal{A}))$) denotes the Auslander–Reiten (AR) quiver of $\mathcal{A}$ (resp. $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$).

2.1. **Persistence modules.** We call each $M \in \operatorname{rep}_{\Bbbk} A_n$, each $N \in \operatorname{rep}_{\Bbbk} A_n(z)$, and each $L \in \operatorname{rep}_{\Bbbk} A_n(a)$ a *persistence module*, a *purely zigzag persistence module*, and a *zigzag persistence module*, respectively. In this subsection, we will define the internal morphisms of an ordinary persistence module and an endofunctor of the category of ordinary persistence modules in order to define the interleaving distance.

For any A_n-type quiver $A_n(a)$, $\alpha_{x,y}$ denotes the arrow between x and y with $1 \leq x < y \leq n$. Then the equioriented A_n-type quiver A_n is

$$A_n : 1 \xrightarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \cdots \xrightarrow{\alpha_{n-1,n}} n \tag{2.5}$$

and a persistence module M has the form

$$M_1 \xrightarrow{M_{\alpha_{1,2}}} M_2 \xrightarrow{M_{\alpha_{2,3}}} \cdots \xrightarrow{M_{\alpha_{n-1,n}}} M_n. \tag{2.6}$$

Moreover, when n is odd, the purely zigzag A_n-type quiver $A_n(z_1)$ is

$$1 \xleftarrow{\alpha_{1,2}} 2 \xrightarrow{\alpha_{2,3}} \cdots \xrightarrow{\alpha_{n-1,n}} n \tag{2.7}$$

and a purely zigzag persistence module $M \in \operatorname{rep}_{\Bbbk} A_n(z_1)$ has the form

$$M_1 \xleftarrow{M_{\alpha_{1,2}}} M_2 \xrightarrow{M_{\alpha_{2,3}}} \cdots \xrightarrow{M_{\alpha_{n-1,n}}} M_n. \tag{2.8}$$

In other cases, we can similarly express the zigzag A_n-type quivers and the zigzag persistence modules.

Definition 4. Let M, N be persistence modules and δ an integer.
(1) For $1 \leq s \leq t \leq n$, the linear map $\phi_M(s,t)\colon M_s \to M_t$ is defined by

$$\phi_M(s,t) = \begin{cases} \mathbb{1}_{M_s}, & s = t \\ M_{\alpha_{t-1,t}} \circ \cdots \circ M_{\alpha_{s,s+1}}, & \text{otherwise} \end{cases}. \tag{2.9}$$

By definition, we have $\phi_M(s,t) = \phi_M(t-1,t) \circ \cdots \circ \phi_M(s,s+1)$.
(2) The δ-shift $M(\delta)$ of M is defined by

$$(M(\delta))_x = \begin{cases} M_{x+\delta}, & 1 \leq x+\delta \leq n \\ 0, & \text{otherwise} \end{cases} \tag{2.10}$$

and

$$(M(\delta))_{\alpha_{x,x+1}} = \begin{cases} M_{\alpha_{x+\delta,x+1+\delta}}, & 1 \leq x+\delta \leq x+1+\delta \leq n \\ 0, & \text{otherwise} \end{cases} \tag{2.11}$$

for each vertex x of A_n. For a morphism $f\colon M \to N$ in $\operatorname{rep}_{\Bbbk} A_n$, the δ-shift $f(\delta)$ of f is defined by

$$(f(\delta))_x = \begin{cases} f_{x+\delta}, & 1 \leq x+\delta \leq n \\ 0, & \text{otherwise} \end{cases} \tag{2.12}$$

for each vertex x of A_n. This defines the *δ-shift functor* $(\delta)\colon \operatorname{rep}_{\Bbbk} A_n \to \operatorname{rep}_{\Bbbk} A_n$. It should be noted that the δ-shift functor can only be defined in the equioriented setting.
(3) Assume that δ is non-negative. The *transition morphism* $\phi_M^\delta\colon M \to M(\delta)$ in $\operatorname{rep}_{\Bbbk} A_n$ is defined by $(\phi_M^\delta)_x = \phi_M(x, x+\delta)$ for each vertex x of A_n. For any morphism $f\colon M \to N$, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi_M^\delta} & M(\delta) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f(\delta)} \\ N & \xrightarrow[\phi_N^\delta]{} & N(\delta). \end{array} \tag{2.13}$$

This defines a natural transformation $\phi^\delta\colon \mathbb{1} \to (\delta)$ from the identity functor $\mathbb{1}$ to the δ-shift functor (δ).
(4) A persistence module M is *δ-trivial* if the transition morphism $\phi_M^\delta\colon M \to M(\delta)$ is zero.

In our setting, the functor (δ) is not an equivalence but an exact functor. Indeed, let M, N, L be persistence modules. A sequence

$$0 \to M \to N \to L \to 0 \tag{2.14}$$

is exact if and only if the sequence

$$0 \to M_x \to N_x \to L_x \to 0 \tag{2.15}$$

is exact for each vertex x of A_n. This means that the sequence

$$0 \to M(\delta) \to N(\delta) \to L(\delta) \to 0 \tag{2.16}$$

is exact.

2.2. **Persistence diagrams and AR quivers.** We recall that the category $\mathrm{rep}_{\Bbbk} A_n(a)$ of zigzag persistence modules is a Krull-Schmidt category, i.e., a representation of $A_n(a)$ is isomorphic to a direct sum of indecomposable representations. In this subsection, we discuss all indecomposable representations of $A_n(a)$.

Definition 5. For $1 \le b \le d \le n$, the *interval representation* $\mathbb{I}[b,d] \in \mathrm{rep}_{\Bbbk} A_n(a)$ is defined by

$$(\mathbb{I}[b,d])_x := \begin{cases} \Bbbk, & b \le x \le d \\ 0, & \text{otherwise} \end{cases} \tag{2.17}$$

and

$$(\mathbb{I}[b,d])_{\alpha_{x,y}} := \begin{cases} \mathbb{1}_{\Bbbk}, & b \le x < y \le d \\ 0, & \text{otherwise} \end{cases}. \tag{2.18}$$

Any interval representation is indecomposable. The converse also holds as follows.

Theorem 6 ([12])**.** *Any indecomposable representation of $A_n(a)$ is isomorphic to an interval representation $\mathbb{I}[b,d]$ for some $1 \le b \le d \le n$.*

Thus, for a representation M of $A_n(a)$, we obtain the unique interval decomposition

$$M \cong \bigoplus_{1 \le b \le d \le n} \mathbb{I}[b,d]^{m(b,d)}, \tag{2.19}$$

leading to the definition of the *persistence diagram* $\mathcal{B}(M)$ of M by

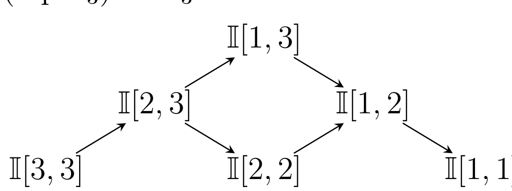
$$\{(b,d,m) \mid 1 \le b \le d \le n, 1 \le m \le m(b,d) \text{ such that } m(b,d) \ne 0\}. \tag{2.20}$$

For simplicity, write an element (b,d,m) of $\mathcal{B}(M)$ as $\langle b,d \rangle$, which is called an *interval.*

From the perspective of AR theory, the persistence diagram of a representation M of $A_n(a)$ can be defined as a map $\Gamma_0 \to \mathbb{Z}$ sending an interval $\mathbb{I}[b,d]$ to its multiplicity $m(b,d)$ in the decomposition of M, where Γ_0 is the set of all interval representations. Note that Γ_0 is the set of vertices of the AR quiver of $A_n(a)$, and in this sense, AR quivers are hidden behind persistence diagrams.

Example 7. The AR quiver $\Gamma(\mathrm{rep}\, A_3)$ of A_3 is

(2.21) $\Gamma(\mathrm{rep}\, A_3) =$



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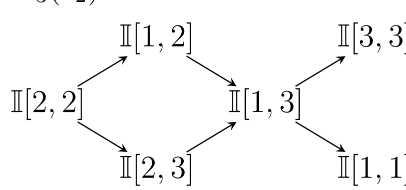
while the AR quiver $\Gamma(\mathrm{rep}\, A_3(z_2))$ of $A_3(z_2) : 1 \to 2 \leftarrow 3$ is

(2.22) $\Gamma(\mathrm{rep}\, A_3(z_2)) =$



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3. Algebraic stability theorem for persistence modules

In this section, we will prove the following Algebraic stability theorem (AST) for persistence modules by using the Induced Matching theorem (IMT) following the paper [1].

Theorem 8 (AST [8],[1])**.** *Let M, N be persistence modules in $\operatorname{rep}_{\Bbbk} A_n$. Then*

$$d_B(\mathcal{B}(M), \mathcal{B}(N)) \leq d_I(M, N). \tag{3.1}$$

where d_I and d_B are the interleaving and the bottleneck distances, respectively.

To prove this, we extend representations M in $\operatorname{rep}_{\Bbbk} A_n$ to those in $\operatorname{rep}_{\Bbbk} A_\ell$ for $\ell \geq n$ as

$$0 \to \cdots \to 0 \to M_1 \to \cdots \to M_n \to 0 \to \cdots \to 0 \in \operatorname{rep}_{\Bbbk} A_\ell. \tag{3.2}$$

Moreover, for a given representation $M \in \operatorname{rep}_{\Bbbk} A_n$ and non-negative integer δ, the map $r^\delta_M \colon \mathcal{B}(M(\delta)) \to \mathcal{B}(M)$ is defined by $r^\delta_M\langle b, d\rangle := \langle b+\delta, d+\delta\rangle$. In general, the map r^δ_M is not bijective. However, we can take an integer $\ell \geq n$ large enough such that r^δ_M is bijective for a given representation M.

3.1. Distances.

First, let us recall the interleaving distance between persistence modules.

Definition 9. Let δ be a non-negative integer. Two persistence modules M and N are said to be *δ-interleaved* if there exist morphisms $f \colon M \to N(\delta)$ and $g \colon N \to M(\delta)$ such that the following diagrams commute:

$$\begin{array}{ccccc} M & & \xrightarrow{\phi^{2\delta}_M} & & M(2\delta), \\ & \searrow^{f} & & \nearrow_{g(\delta)} & \\ & & N(\delta) & & \end{array} \qquad \begin{array}{ccccc} N & & \xrightarrow{\phi^{2\delta}_N} & & N(2\delta). \\ & \searrow^{g} & & \nearrow_{f(\delta)} & \\ & & M(\delta) & & \end{array} \tag{3.3}$$

In this case, we call the pair of $f \colon M \to N(\delta)$ and $g \colon N \to M(\delta)$ a *δ-interleaving pair*. Moreover, we call a morphism $f \colon M \to N(\delta)$ a *δ-interleaving morphism* if there is a morphism $g \colon N \to M(\delta)$ such that the pair (f, g) is a δ-interleaving pair.

For persistence modules M, N, the *interleaving distance* is defined as

$$d_I(M, N) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid M \text{ and } N \text{ are } \delta\text{-interleaved}\}. \tag{3.4}$$

We remark that in our setting, $d_I(M, N) = 0$ if and only if M and N are isomorphic. Thus, the interleaving distance measures how far these modules are from being isomorphic.

We next recall the bottleneck distance between persistence diagrams: a *matching* from a set S to a set T (written as $\sigma \colon S \nrightarrow T$) is a bijection $\sigma \colon S' \to T'$ for some subset S' of S and some subset T' of T. For such a matching σ, we write S' as $\operatorname{Coim}\sigma$ and T' as $\operatorname{Im}\sigma$.

For totally ordered sets, a matching can be defined canonically as follows: let $S = \{S_i \mid i = 1, \cdots, s\}$ and $T = \{T_i \mid i = 1, \cdots, t\}$ be finite totally ordered sets such that for $a \leq b$, $S_a \leq S_b$ and $T_a \leq T_b$. Then a *canonical matching* $\sigma \colon S \nrightarrow T$ is a matching σ given by $\sigma(S_i) = T_i$ for $i = 1, \cdots, \min\{s, t\}$. In this case, either $\operatorname{Im}\sigma = S$ or $\operatorname{Coim}\sigma = T$ is satisfied.

Definition 10. Let δ be a non-negative integer. For a persistence diagram $\mathcal{B}$, let $\mathcal{B}_\delta$ be the subset of $\mathcal{B}$ consisting of intervals $\langle b, d\rangle$ such that $d - b \geq \delta$. A *δ-matching* between persistence diagrams $\mathcal{B}$ and $\mathcal{B}'$ is defined by a matching $\sigma\colon \mathcal{B} \nrightarrow \mathcal{B}'$ such that

$$\mathcal{B}_{2\delta} \subseteq \operatorname{Coim}\sigma, \quad \mathcal{B}'_{2\delta} \subseteq \operatorname{Im}\sigma \tag{3.5}$$

and for all $\sigma\langle b, d\rangle = \langle b', d'\rangle$,

$$b' - \delta \leq b \leq d \leq d' + \delta, \quad b - \delta \leq b' \leq d' \leq d + \delta. \tag{3.6}$$

Two persistence diagrams $\mathcal{B}$ and $\mathcal{B}'$ are said to be *δ-matched* if there is a δ-matching between $\mathcal{B}$ and $\mathcal{B}'$. The *bottleneck distance* between $\mathcal{B}$ and $\mathcal{B}'$ is defined as

$$d_B(\mathcal{B}, \mathcal{B}') := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid \mathcal{B} \text{ and } \mathcal{B}' \text{ are } \delta\text{-matched}\}. \tag{3.7}$$

Note that equation (3.6) implies that the interval representations associated with $\langle b, d\rangle$, $\langle b', d'\rangle$ are δ-interleaved.

We will extend this concept to the derived setting later (see Definition 18 and Definition 21).

3.2. **Proof of AST by IMT.** Here, we will explain the proof of an AST for $\operatorname{rep}_{\Bbbk} A_n$ following [1]. Their strategy utilizes an IMT.

Let M be a persistence module. For $1 \leq b \leq n$, $\mathcal{B}(M)_{\langle b,\text{-}\rangle}$ denotes the subset of $\mathcal{B}(M)$ consisting of the intervals $\langle b, c\rangle$ for some $b \leq c \leq n$, and $\mathcal{B}(M)_{\langle \text{-},d\rangle}$ denotes the subset of $\mathcal{B}(M)$ consisting of the intervals $\langle c, d\rangle$ for some $1 \leq c \leq d$. Note that $\mathcal{B}(M)_{\langle b,\text{-}\rangle}$ and $\mathcal{B}(M)_{\langle \text{-},d\rangle}$ are regarded as totally ordered sets with the total order induced by the reverse inclusion relation on intervals. Indeed, if $c < c'$, then $\langle b, c'\rangle < \langle b, c\rangle$ in $\mathcal{B}(M)_{\langle b,\text{-}\rangle}$ and $\langle c, d\rangle < \langle c', d\rangle$ in $\mathcal{B}(M)_{\langle \text{-},d\rangle}$.

Definition 11. Let $f\colon M \to N$ be a morphism in $\operatorname{rep}_{\Bbbk} A_n$. Then the *induced matching* $\mathcal{B}(f)\colon \mathcal{B}(M) \to \mathcal{B}(N)$ is defined as follows:

(1) When f is injective, $\mathcal{B}(f)$ is defined via the family of canonical matchings from $\mathcal{B}(M)_{\langle \text{-},d\rangle}$ to $\mathcal{B}(N)_{\langle \text{-},d\rangle}$.
(2) When f is surjective, $\mathcal{B}(f)$ is defined via the family of canonical matchings from $\mathcal{B}(M)_{\langle b,\text{-}\rangle}$ to $\mathcal{B}(N)_{\langle b,\text{-}\rangle}$.
(3) Any morphism f can be decomposed into the surjective morphism $\pi\colon M \to \operatorname{Im} f$ and the injective morphism $\mu\colon \operatorname{Im} f \to N$. Then $\mathcal{B}(f) := \mathcal{B}(\mu) \circ \mathcal{B}(\pi)$ by (1) and (2).

This matching is what yields the IMT (see [1, Theorem 4.2]).

Theorem 12 (IMT)**.** *Let $f\colon M \to N$ be a morphism in* $\operatorname{rep}_{\Bbbk} A_n$*. Assume that* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *are 2δ-trivial. Moreover, taking an integer $\ell \geq n$ large enough such that r^δ_M is bijective, one regards M, N as representations of A_ℓ. Then $\mathcal{B}(f) \circ r^\delta_M$ is a δ-matching $\mathcal{B}(M(\delta)) \nrightarrow \mathcal{B}(N)$.*

Let $f\colon M \to N(\delta)$ be a δ-interleaving morphism. It is easily seen that $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are 2δ-trivial. Thus, Theorem 12 induces Theorem 8 as follows.

Proof of Theorem 8. Let $f\colon M \to N(\delta)$ be a δ-interleaving morphism in $\operatorname{rep}_{\Bbbk} A_n$ and $\ell \geq n$ an integer large enough such that r^{δ}_{M} and r^{δ}_{N} are bijective. Then M and N are regarded as representations of A_ℓ. Since $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are 2δ-trivial,

$$r^{\delta}_{N} \circ \mathcal{B}(f) = r^{\delta}_{N} \circ (\mathcal{B}(f) \circ r^{\delta}_{M}) \circ (r^{\delta}_{M})^{-1}\colon \mathcal{B}(M) \xrightarrow{\sim} \mathcal{B}(M(\delta)) \twoheadrightarrow \mathcal{B}(N(\delta)) \xrightarrow{\sim} \mathcal{B}(N) \tag{3.8}$$

is a δ-matching by Theorem 12, as desired. □

3.3. **Isometry theorem.** Theorem 8 gives the inequality $d_B \leq d_I$, which is a part of the following isometry theorem (see [1, Theorem 3.1 and Section B.1]).

Theorem 13 (Isometry theorem)**.** *Let M, N be persistence modules. Then*

$$d_B(\mathcal{B}(M), \mathcal{B}(N)) = d_I(M, N). \tag{3.9}$$

4. Algebraic stability theorem for derived categories of persistence modules and zigzag ones

It is known that if abelian categories $\mathcal{A}$ and $\mathcal{A}'$ are derived equivalent, their AR quivers $\Gamma(\mathcal{A})$ and $\Gamma(\mathcal{A}')$ are isomorphic. The AR quivers can be regarded as the persistence diagrams in a sense. Thus, we adopt the strategy to get an AST for zigzag persistence modules as follows: let $\mathcal{A}'$ be an abelian category. To get an AST for $\mathcal{A}'$, let $\mathcal{A}$ be an abelian category, derived equivalent to $\mathcal{A}'$, for which an AST holds. Then, we will extend the AST to that for the derived category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Through the derived equivalence $\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A}')$, we obtain an AST for $\mathsf{D}^{\mathrm{b}}(\mathcal{A}')$ and $\mathcal{A}'$ as a subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A}')$.

In this section, we consider that the abelian category $\mathcal{A}'$ is the category of zigzag persistence modules and the abelian category $\mathcal{A}$ is the category of persistence modules.

4.1. **Derived distances.** In this subsection, we propose distances on the derived category of persistence modules by extending the original interleaving and bottleneck distances.

Recall that the δ-shift functor $(\delta)\colon \operatorname{rep}_{\Bbbk} A_n \to \operatorname{rep}_{\Bbbk} A_n$ induces a functor

$$(\delta)\colon \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n) \to \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n) \tag{4.1}$$

via $X^{\bullet}(\delta) = (X^i(\delta), d^i_X(\delta))_{i \in \mathbb{Z}}$ since the functor (δ) is exact.

For $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$, we have the following strong characterization of a cochain complex by its cohomologies.

Lemma 14. *For any cochain complex $X^{\bullet} \in \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$,*

$$X^{\bullet} \cong \bigoplus_{i \in \mathbb{Z}} H^i(X^{\bullet})[-i] \tag{4.2}$$

in $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$. More generally, in the case of $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$,

$$X^{\bullet}(\delta) \cong \bigoplus_{i \in \mathbb{Z}} H^i(X^{\bullet})(\delta)[-i]. \tag{4.3}$$

The fact that the global dimension of $\operatorname{rep}_{\Bbbk} A_n(a)$ is at most 1 is essential to the proof of the foregoing lemma. As a consequence of Lemma 14, we can characterize all indecomposable objects of $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$.

Corollary 15. *A cochain complex $X^\bullet \in \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$ is indecomposable if and only if $X^\bullet$ is isomorphic to a stalk complex*

$$\mathbb{I}[b,d][-i] : \cdots \to 0 \to \mathbb{I}[b,d] \to 0 \to \cdots \tag{4.4}$$

concentrated at the i-th term in $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$ for some $1 \le b \le d \le n$ and some $i \in \mathbb{Z}$. Thus, any cochain complex $X^\bullet$ is isomorphic to

$$\bigoplus_{b \le d, i} (\mathbb{I}[b,d][-i])^{m(b,d,i)}, \tag{4.5}$$

where the non-negative integer $m(b,d,i)$ is the multiplicity of $\mathbb{I}[b,d][-i]$.

Since $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$ is a Krull-Schmidt category (see [9]), the interval decomposition in the corollary above is unique. By using this result, we propose the notion of a 'derived' persistence diagram. Note that Berkouk–Ginot [2] considered a similar concept of a graded persistence diagram.

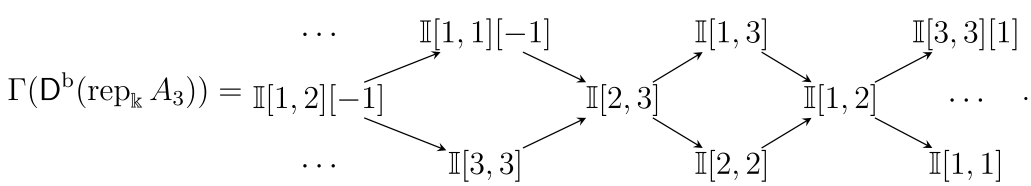
Definition 16. Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$. Then the *derived persistence diagram* $\mathcal{B}^D(X^\bullet)$ is defined as

$$\mathcal{B}^D(X^\bullet) := \bigsqcup_{i \text{ with } H^i(X^\bullet) \neq 0} \mathcal{B}(H^i(X^\bullet)) \tag{4.6}$$

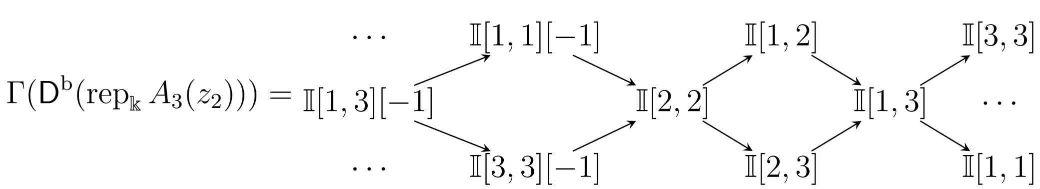
where $\mathcal{B}(H^i(X^\bullet))$ is the ordinary persistence diagram of $H^i(X^\bullet)$.

Similar to the case of $\operatorname{rep}_{\Bbbk} A_n(a)$, the derived persistence diagram of $X^\bullet$ can be defined as a map $\Gamma_0 \to \mathbb{Z}$ sending $\mathbb{I}[b,d][-i]$ to the multiplicity $m(b,d,i)$, where Γ_0 is the set of vertices of the AR quiver of $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$. Thus, AR quivers are hidden behind persistence diagrams also in this setting. Moreover, the AR quiver of $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$ consists of all shifted copies of the AR quiver $\Gamma(A_n(a))$ of $A_n(a)$.

Example 17. The AR quiver $\Gamma(\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_3))$ is



Moreover, the AR quiver $\Gamma(\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_3(z_2)))$, where $A_3(z_2) : 1 \to 2 \leftarrow 3$, is

(4.7)


Here, we define the derived interleaving distance.

Definition 18. Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$ and δ a non-negative integer. Then $X^\bullet$ and $Y^\bullet$ are said to be *derived δ-interleaved* if there exist morphisms $f^\bullet \colon X^\bullet \to Y^\bullet(\delta)$ and $g^\bullet \colon Y^\bullet \to X^\bullet(\delta)$ such that for each $i \in \mathbb{Z}$, $(H^i(f^\bullet), H^i(g^\bullet))$ is a

δ-interleaving pair between $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ in the sense of Definition 9. Namely, the following diagrams commute for each $i \in \mathbb{Z}$:

$$\begin{array}{ccccccc} H^i(X^\bullet) & \xrightarrow{\phi^{2\delta}_{H^i(X^\bullet)}} & H^i(X^\bullet)(2\delta), & \quad & H^i(Y^\bullet) & \xrightarrow{\phi^{2\delta}_{H^i(Y^\bullet)}} & H^i(Y^\bullet)(2\delta) . \\ & {\scriptstyle H^i(f^\bullet)} \searrow \quad \nearrow {\scriptstyle H^i(g^\bullet)(\delta)} & & & & {\scriptstyle H^i(g^\bullet)} \searrow \quad \nearrow {\scriptstyle H^i(f^\bullet)(\delta)} & \\ & H^i(Y^\bullet)(\delta) & & & & H^i(X^\bullet)(\delta) & \end{array} \tag{4.8}$$

In this case we also call the pair of $f^\bullet \colon X^\bullet \to Y^\bullet(\delta)$ and $g^\bullet \colon Y^\bullet \to X^\bullet(\delta)$ a *derived δ-interleaving pair*. Moreover, we call a morphism $f^\bullet \colon X^\bullet \to Y^\bullet(\delta)$ a *derived δ-interleaving morphism* if there is a morphism $g^\bullet \colon Y^\bullet \to X^\bullet(\delta)$ such that the pair $(f^\bullet, g^\bullet)$ is a derived δ-interleaving pair.

For cochain complexes $X^\bullet, Y^\bullet$ in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$, the *derived interleaving distance* is defined as

$$d_I^D(X^\bullet, Y^\bullet) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid X^\bullet \text{ and } Y^\bullet \text{ are derived } \delta\text{-interleaved}\}. \tag{4.9}$$

Remark 19. Similarly to the original setting, $d_I^D(X^\bullet, Y^\bullet) = 0$ for two cochain complexes $X^\bullet, Y^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$ if and only if $X^\bullet$ and $Y^\bullet$ are isomorphic in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$. Thus, the derived interleaving distance also measures how far these complexes are from being isomorphic.

Note that if $X^\bullet$ and $Y^\bullet$ are derived δ-interleaved, then $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are δ-interleaved for all i. It follows from Lemma 14 that the converse also holds.

Corollary 20. *Let $X^\bullet$, $Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$. Then $X^\bullet$ and $Y^\bullet$ are derived δ-interleaved if and only if $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are δ-interleaved for all i.*

Finally, we propose the 'derived' bottleneck distance between derived persistence diagrams in the sense of Definition 16 in this setting.

Definition 21. Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$. Two derived persistence diagrams $\mathcal{B}^D(X^\bullet)$ and $\mathcal{B}^D(Y^\bullet)$ are said to be *δ-matched* if $B(H^i(X^\bullet))$ and $B(H^i(Y^\bullet))$ are δ-matched in the sense of Definition 10 for all $i \in \mathbb{Z}$.

For derived persistence diagrams $\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)$, the *derived bottleneck distance* is defined as

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid \mathcal{B}^D(X^\bullet) \text{ and } \mathcal{B}^D(Y^\bullet) \text{ are } \delta\text{-matched}\}. \tag{4.10}$$

4.2. **AST for derived categories.** In this subsection, we first prove an AST for derived categories of persistence modules.

Theorem 22 (AST for derived categories)**.** *Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$. Then*

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) \leq d_I^D(X^\bullet, Y^\bullet). \tag{4.11}$$

Proof. Assume that $X^\bullet$ and $Y^\bullet$ are derived δ-interleaved. Then for all $i \in \mathbb{Z}$, $H^i(X^\bullet)$ and $H^i(Y^\bullet)$ are δ-interleaved, and hence $\mathcal{B}(H^i(X^\bullet))$ and $\mathcal{B}(H^i(Y^\bullet))$ are δ-matched by Theorem 8. Thus, by definition, the inequality

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) \leq d_I^D(X^\bullet, Y^\bullet) \tag{4.12}$$

holds. ∎

Next, we consider a distance induced by a derived equivalence. We can assume that there exists a derived equivalence E from $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ to $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$ with $\mathcal{A} = \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n(a))$. For example, let $T \in \operatorname{rep}_{\Bbbk} A_n$ be a classical tilting module and take $E^{-1} = \mathrm{RHom}(T, \text{-})$ (see [14, Chapter III]).

Definition 23. Two objects X and Y of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$ are said to be *δ-interleaved with respect to E* if $E(X)$ and $E(Y)$ are derived δ-interleaved in the sense of Definition 18. The *interleaving distance $d_I^{E,\mathcal{A}}(X,Y)$ with respect to E* is defined as

$$d_I^{E,\mathcal{A}}(X,Y) := \inf\{\delta \in \mathbb{Z}_{\geq 0} \mid X \text{ and } Y \text{ are } \delta\text{-interleaved with respect to } E\}. \tag{4.13}$$

Namely, $d_I^{E,\mathcal{A}}(X,Y) = d_I^D(E(X), E(Y))$ holds.

Remark 24. By Remark 19, $d_I^{E,\mathcal{A}}(X,Y) = 0$ if and only if $E(X)$ and $E(Y)$ are isomorphic in $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$. Since E is an equivalence, this means that X and Y are isomorphic in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Thus, the interleaving distance defined as above also measures how far these objects are from being isomorphic. This justifies calling the distance an interleaving distance.

Remark 25. The δ-shift functor cannot be defined in the zigzag setting, so neither can the usual interleaving distance. One of the advantages of our approach is that we can define the interleaving distance even in the zigzag setting through the derived equivalence.

Since E is an equivalence, in particular, a fully faithful functor, $X \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is indecomposable if and only if so is $E(X) \in \mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$. Hence, since $\mathsf{D}^{\mathrm{b}}(\operatorname{rep}_{\Bbbk} A_n)$ is a Krull-Schmidt category, so is $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Consequently, the derived equivalence E induces a bijection between $\mathcal{B}^D(E(X))$ (see Definition 16) and

$$\mathcal{B}^D_{\mathcal{A}}(X) := \{Z \in \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \mid Z \text{ is indecomposable and a direct summand of } X\}. \tag{4.14}$$

Then the following distance between $\mathcal{B}^D_{\mathcal{A}}(X)$ and $\mathcal{B}^D_{\mathcal{A}}(Y)$ is naturally derived by passing through the derived equivalence E.

Definition 26. For two objects X, Y of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, $\mathcal{B}^D_{\mathcal{A}}(X)$ and $\mathcal{B}^D_{\mathcal{A}}(Y)$ are said to be *δ-matched with respect to E* if $\mathcal{B}^D(E(X))$ and $\mathcal{B}^D(E(Y))$ are δ-matched in the sense of Definition 21. The *bottleneck distance $d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y))$ with respect to E* is defined as

$$d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y)) := \inf\left\{\delta \in \mathbb{Z}_{\geq 0} \;\middle|\; \begin{array}{c} \mathcal{B}^D_{\mathcal{A}}(X) \text{ and } \mathcal{B}^D_{\mathcal{A}}(Y) \text{ are} \\ \delta\text{-matched with respect to } E \end{array}\right\}. \tag{4.15}$$

Namely, $d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y)) = d_B^D(\mathcal{B}^D(E(X)), \mathcal{B}^D(E(Y)))$ holds.

In our convention, *an AST* states that the interleaving distance between objects X and Y gives an upper bound for the bottleneck distance between their persistence diagrams. Thus, as a consequence of Theorem 22, Definition 23, and Definition 26, we have the following AST for the derived category $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

Proposition 27 ([15])**.** *Let X, Y objects in $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Then*

$$d_B^{E,\mathcal{A}}(\mathcal{B}^D_{\mathcal{A}}(X), \mathcal{B}^D_{\mathcal{A}}(Y)) \leq d_I^{E,\mathcal{A}}(X,Y). \tag{4.16}$$

Recall that $\mathcal{A}$ can be regarded as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. As a consequence of Proposition 27, an AST holds for zigzag persistence modules. Thus, we obtain an AST for a wider class compared to that of Botnan and Lesnick [6].

4.3. **Isometry theorem for derived categories.** By Theorem 13 and Theorem 22, we obtain an isometry theorem for the derived category of persistence modules.

Theorem 28 (Isometry theorem for derived categories)**.** *Let $X^\bullet, Y^\bullet$ be cochain complexes in $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} A_n)$. Then*

$$d_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet)) = d_I^D(X^\bullet, Y^\bullet). \tag{4.17}$$

As a consequence of Theorem 28, we can extend Proposition 27 to isometry theorems by Definition 23 and Definition 26.

Corollary 29. *Let X, Y objects in $\mathcal{A}$ or $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$. Then*

$$d_B^{E,\mathcal{A}}(B_{\mathcal{A}}^D(X), B_{\mathcal{A}}^D(Y)) = d_I^{E,\mathcal{A}}(X, Y). \tag{4.18}$$

The special case of Corollary 29 is exactly an isometry theorem for purely zigzag persistence modules.

5. Comparison among distances

5.1. **Block distance.** Botnan–Lesnick [6] proved an AST for purely zigzag persistence modules. In that paper, they introduced the interleaving and bottleneck distances on purely zigzag persistence modules. Bjerkevik [4] proved that those distances actually coincide. Here, we refer to the interleaving distance as the *block distance*, denoted by d_{BL}, following the paper [19].

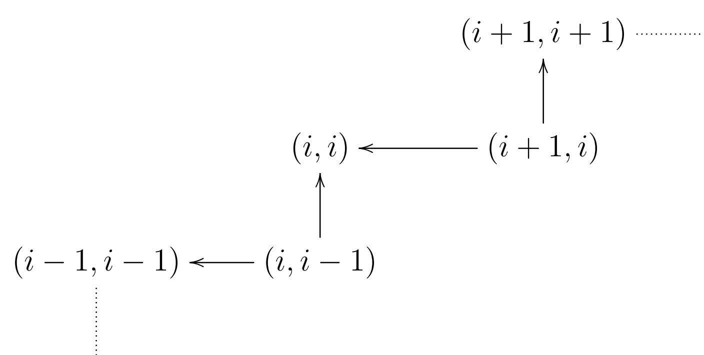
First, we explain the block distance defined by Botnan–Lesnick [6]. For this aim, we will introduce the infinite purely zigzag quiver $\mathbb{ZZ}$.

Let $\mathbb{Z}$ be the poset of integers with usual order and $\mathbb{Z}^{\mathrm{op}}$ its opposite poset. As in [6], let $\mathbb{ZZ}$ be the subposet of the poset $\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}$ given by

$$\mathbb{ZZ} := \{(i, j) \mid i \in \mathbb{Z}, j \in \{i, i-1\}\}. \tag{5.1}$$

Note that this can be expressed by the infinite purely zigzag quiver

(5.2) $Q =$



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so that $\mathbb{ZZ}$ and Q are identified and a (pointwise finite-dimensional) representation of $\mathbb{ZZ}$ is just that of the quiver Q. We use $\mathrm{rep}_{\Bbbk} \mathbb{ZZ}$ to denote the category of representations of $\mathbb{ZZ}$.

Moreover, in [6], the intervals $\langle b,d\rangle_{\mathbb{ZZ}}$ $(b \leq d)$ of $\mathbb{ZZ}$ are classified into the following 4 types:

$$(5.3)\qquad \begin{cases} \text{closed interval} & [b,d]_{\mathbb{ZZ}} := \{(i,j) \in \mathbb{ZZ} \mid (b,b) \leq (i,j) \leq (d,d)\}, \\ \text{right-open interval} & [b,d)_{\mathbb{ZZ}} := \{(i,j) \in \mathbb{ZZ} \mid (b,b) \leq (i,j) < (d,d)\}, \\ \text{left-open interval} & (b,d]_{\mathbb{ZZ}} := \{(i,j) \in \mathbb{ZZ} \mid (b,b) < (i,j) \leq (d,d)\}, \\ \text{open interval} & (b,d)_{\mathbb{ZZ}} := \{(i,j) \in \mathbb{ZZ} \mid (b,b) < (i,j) < (d,d)\}. \end{cases}$$

We use $\mathbb{I}^{\langle b,d\rangle_{\mathbb{ZZ}}}$ to denote the interval representation of $\mathbb{ZZ}$ associated with the interval $\langle b,d\rangle_{\mathbb{ZZ}}$. Note that the interval representation $\mathbb{I}^{\langle b,d\rangle_{\mathbb{ZZ}}}$ of $\mathbb{ZZ}$ is uniquely determined by the interval $\langle b,d\rangle_{\mathbb{ZZ}}$. Indeed, $\mathbb{I}^{\langle b,d\rangle_{\mathbb{ZZ}}}$ is the representation given by

$$(5.4)\qquad \mathbb{I}^{\langle b,d\rangle_{\mathbb{ZZ}}}_{(i,j)} = \begin{cases} \Bbbk, & (i,j) \in \langle b,d\rangle_{\mathbb{ZZ}} \\ 0, & \text{otherwise} \end{cases}$$

and is called a *closed* (resp. *right-open, left-open,* and *open*) interval representation if $\langle b,d\rangle_{\mathbb{ZZ}}$ is closed (resp. right-open, left-open, and open). Note that interval representations are indecomposable and every pointwise finite representation of $\mathbb{ZZ}$ can be decomposed into interval representations (see [5]).

The distance d_{BL} is defined via the interleaving distance on 2D persistence modules. For that purpose, Botnan–Lesnick [6] defined an embedding functor $J\colon \operatorname{rep}_{\Bbbk} \mathbb{ZZ} \to \operatorname{rep}_{\Bbbk} \mathbb{U}$, where $\mathbb{U} := \{(a,b) \in \mathbb{R}^2 \mid b \geq a\}$ and $\operatorname{rep}_{\Bbbk} \mathbb{U}$ is the category of representations of $\mathbb{U}$. The functor J was originally denoted by E in loc. cit. Note that $\mathbb{U}$ is a subposet of $\mathbb{R}^{\mathrm{op}} \times \mathbb{R}$ and that a poset can be expressed by a quiver Q' with relations in general. Hence, a representation of $\mathbb{U}$ is a representation of the quiver Q' satisfying the condition induced from the relations. For $\varepsilon \geq 0$, we set $\vec{\varepsilon} := (-\varepsilon, \varepsilon) \in \mathbb{R}^2_{\geq 0}$ and define a shift functor $[\vec{\varepsilon}]\colon \operatorname{rep}_{\Bbbk} \mathbb{U} \to \operatorname{rep}_{\Bbbk} \mathbb{U}$ on objects by $M[\vec{\varepsilon}]_u := M_{u+\vec{\varepsilon}}$ together with natural morphisms. For $M \in \operatorname{rep}_{\Bbbk} \mathbb{U}$ and $\varepsilon \geq 0$, there is a canonical morphism $\phi^{\vec{\varepsilon}}_M\colon M \to M[\vec{\varepsilon}]$.

Definition 30. (1) For $M, N \in \operatorname{rep}_{\Bbbk} \mathbb{U}$ and $\varepsilon \geq 0$, M and N are said to be *ε-interleaved* if there exist morphisms $f\colon M \to N[\vec{\varepsilon}]$ and $g\colon M \to N[\vec{\varepsilon}]$ such that the following diagrams commute:

$$(5.5)\qquad \begin{array}{ccc} M & \xrightarrow{\phi^{2\vec{\varepsilon}}_M} & M[2\vec{\varepsilon}] \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g[\vec{\varepsilon}]} & \\ & N[\vec{\varepsilon}] & \end{array}, \qquad \begin{array}{ccc} N & \xrightarrow{\phi^{2\vec{\varepsilon}}_N} & N[2\vec{\varepsilon}] \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle f[\vec{\varepsilon}]} & \\ & M[\vec{\varepsilon}] & \end{array}.$$

The *interleaving distance on* $\operatorname{rep}_{\Bbbk} \mathbb{U}$ is defined as

$$(5.6)\qquad d^{\mathbb{U}}_I(M,N) := \inf\{\varepsilon \geq 0 \mid M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}.$$

(2) The *block distance* d_{BL} on $\operatorname{rep}_{\Bbbk} \mathbb{ZZ}$ is defined as

$$(5.7)\qquad d_{BL}(X,Y) := d^{\mathbb{U}}_I(J(X), J(Y)).$$

The name of block distance will be justified below in this subsection.

We recall some properties of J, for which we need blocks and block-decomposable representations (see Section 3 in [6] or Definitions 2.5 and 2.10 in [3]).

Definition 31. A *block* B is a subset of $\mathbb{R}^2$ of the following type:

(1) A *birthblock* (**bb** for short) if there is $(a, b) \in \mathbb{R}^2$ such that $B = \langle -\infty, a\rangle \times \langle b, +\infty\rangle$, where (a, b) can be $(+\infty, -\infty)$. Moreover, B is said to be of type $\mathbf{bb}^+$ if $b > a$, and $\mathbf{bb}^-$ otherwise.
(2) A *deathblock* (**db** for short) if there is $(a, b) \in \mathbb{R}^2$ such that $B = \langle a, +\infty\rangle \times \langle -\infty, b\rangle$. Moreover, B is said to be of type $\mathbf{db}^+$ if $b > a$, and $\mathbf{db}^-$ otherwise.
(3) A *horizontalblock* (**hb** for short) if there is $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that $B = \mathbb{R} \times \langle a, b\rangle$.
(4) A *verticalblock* (**vb** for short) if there is $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R}$ such that $B = \langle a, b\rangle \times \mathbb{R}$.

Using the above definition, we define block representations and block-decomposable representations of $\mathbb{U}$.

Definition 32. (1) A *block representation M of type B* of $\mathbb{U}$ is defined by, for $x \leq y \in \mathbb{U}$,

$$M(x) = \begin{cases} \Bbbk, & x \in \mathbb{U} \cap B \\ 0, & \text{otherwise} \end{cases} \text{ and } M(x \leq y) = \begin{cases} \mathbb{1}_{\Bbbk}, & x, y \in \mathbb{U} \cap B \\ 0, & \text{otherwise} \end{cases}. \tag{5.8}$$

Note that any block representation is indecomposable.
(2) A representation M of $\mathbb{U}$ is called *block-decomposable* if M can be only decomposed into block representations.

Remark that a block representation can be 0 when the corresponding block is of type $\mathbf{db}^-$.

The functor J sends an object of $\mathrm{rep}_{\Bbbk}\,\mathbb{ZZ}$ to a block-decomposable persistence module. In fact, each type of interval module is sent as follows.

Lemma 33 (Lemma 4.1 in [6])**.** *The functor J sends closed, open, right-open, and left-open interval representations to block representations of type* $\mathbf{bb}^-, \mathbf{db}^+, \mathbf{hb}$*, and* $\mathbf{vb}$*, respectively.*

5.2. **Convolution distance.** In this subsection, we study the convolution distance introduced by Kashiwara–Schapira [18]. The convolution distance is defined as a distance on the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$ of constructible sheaves on $\mathbb{R}$ (indeed in a more general setting).

First, we prove the category of representations of the infinite zigzag quiver $\mathbb{ZZ}^{\mathrm{op}}$ is equivalent to some sheaf category on $\mathbb{R}$. The equivalence induces a distance on the derived category of such representations.

Let us briefly recall the notion of sheaves and fix some notation. Let X be a topological space and $\mathrm{Open}(X)$ the category of open subsets of X whose Hom-set $\mathrm{Hom}(U, V)$ is the singleton if $U \subset V$ and empty otherwise. A sheaf F of $\Bbbk$-vector spaces on X is a functor $\mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Vect}(\Bbbk)$ with some gluing condition (see [17] for example). We write $\mathrm{Sh}(\Bbbk_X)$ for the abelian category of sheaves of $\Bbbk$-vector spaces on X. In what follows, we focus on sheaves on $\mathbb{R}$. A sheaf $F \in \mathrm{Sh}(\Bbbk_{\mathbb{R}})$ is said to be *constructible* if there exist discrete points $\{x_k\}_{k\in\mathbb{Z}}$ with $x_k < x_{k+1}$ such that $F|_{(x_k, x_{k+1})}$ is locally constant for any $k \in \mathbb{Z}$ and F_t is finite-dimensional for any $t \in \mathbb{R}$. We denote by $\mathrm{Sh}_c(\Bbbk_{\mathbb{R}})$ the full subcategory of $\mathrm{Sh}(\Bbbk_{\mathbb{R}})$ consisting of constructible sheaves.

Definition 34. One defines $\mathrm{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})$ as the full subcategory of $\mathrm{Sh}_c(\Bbbk_{\mathbb{R}})$ consisting of objects F such that $F|_{(i,i+1)}$ is constant for any $i \in \mathbb{Z}$.

Now we consider the relation between representations of the infinite zigzag quiver and sheaves on $\mathbb{R}$, which is essentially studied by Guillermou [13]. For $X \in \mathrm{rep}_{\Bbbk} \mathbb{Z}\mathbb{Z}^{\mathrm{op}}$, we set

$$F_t^X := \begin{cases} X_{(i,i)}, & t = i \in \mathbb{Z} \\ X_{(i,i-1)}, & i-1 \le t \le i \ (i \in \mathbb{Z}) \end{cases} \tag{5.9}$$

and define $S(X) \in \mathrm{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})$ by

$$S(X)(U) := \left\{ f \in \prod_{t \in U} F_t^X \;\middle|\; \begin{array}{l} f|_{U \setminus \mathbb{Z}} \text{ is locally constant,} \\ \text{for any } y \in \mathbb{Z} \cap U \text{ and } \varepsilon > 0 \text{ small enough,} \\ f(y-\varepsilon) = X_{\beta_y}(f(y)) \text{ with } \beta_y = \alpha_{(y,y-1),(y,y)}, \\ f(y+\varepsilon) = X_{\beta'_y}(f(y)) \text{ with } \beta'_y = \alpha_{(y+1,y),(y,y)} \end{array} \right\}. \tag{5.10}$$

Note that with a morphism $\varphi \colon X \to Y$ one can associate a canonical morphism $S(\varphi) \colon S(X) \to S(Y)$. The correspondence defines a functor $S \colon \mathrm{rep}_{\Bbbk} \mathbb{Z}\mathbb{Z}^{\mathrm{op}} \to \mathrm{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})$. For this functor, we have the following equivalence.

Proposition 35. *The functor $S \colon \mathrm{rep}_{\Bbbk} \mathbb{Z}\mathbb{Z}^{\mathrm{op}} \to \mathrm{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})$ is an equivalence of categories.*

Here we recall the convolution distance on $\mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$. Let $q_1, q_2 \colon \mathbb{R}^2 \to \mathbb{R}$ be the first and second projections. Moreover, we set $s \colon \mathbb{R}^2 \to \mathbb{R}, (t_1, t_2) \mapsto t_1 + t_2$. For $F^\bullet, G^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$, their convolution $F^\bullet \star G^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$ is defined by

$$F^\bullet \star G^\bullet := Rs_!(q_1^{-1} F \otimes q_2^{-1} G). \tag{5.11}$$

For $\varepsilon \ge 0$, we set $K_\varepsilon := \Bbbk_{[-\varepsilon,\varepsilon]} \in \mathrm{Sh}_c(\Bbbk_{\mathbb{R}})$, which has stalks $\Bbbk$ on $[-\varepsilon, \varepsilon]$ and 0 otherwise. We have $K_\varepsilon \star K_{\varepsilon'} \cong K_{\varepsilon+\varepsilon'}$ for $\varepsilon, \varepsilon' \ge 0$. For $\varepsilon \ge 0$, the canonical morphism $K_\varepsilon \to K_0$ induces a morphism $\phi_{F^\bullet}^{\varepsilon} \colon F^\bullet \star K_\varepsilon \to F^\bullet \star K_0 \cong F^\bullet$. For $F^\bullet, G^\bullet \in \mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$ and $\varepsilon \ge 0$, $F^\bullet$ and $G^\bullet$ are said to be *ε-isomorphic* if there exist morphisms $f \colon K_\varepsilon \star F^\bullet \to G^\bullet$ and $g \colon K_\varepsilon \star G^\bullet \to F^\bullet$ such that following diagrams commute:

$$\begin{array}{ccc} F^\bullet \star K_{2\varepsilon} & \xrightarrow{\phi_{F^\bullet}^{2\varepsilon}} & F^\bullet \\ & {\scriptstyle f \star K_\varepsilon} \searrow \quad \nearrow {\scriptstyle g} & \\ & G^\bullet \star K_\varepsilon & \end{array}, \qquad \begin{array}{ccc} G^\bullet \star K_{2\varepsilon} & \xrightarrow{\phi_{G^\bullet}^{2\varepsilon}} & G^\bullet \\ & {\scriptstyle g \star K_\varepsilon} \searrow \quad \nearrow {\scriptstyle f} & \\ & F^\bullet \star K_\varepsilon & \end{array}. \tag{5.12}$$

The *convolution distance* on $\mathsf{D}^{\mathrm{b}}(\mathrm{Sh}_c(\Bbbk_{\mathbb{R}}))$ is defined as

$$d_C(F^\bullet, G^\bullet) := \inf\{\varepsilon \ge 0 \mid F^\bullet \text{ and } G^\bullet \text{ are } \varepsilon\text{-isomorphic}\}. \tag{5.13}$$

Through the equivalence in Proposition 35, the convolution distance induces a distance on $\mathsf{D}^{\mathrm{b}}(\mathrm{rep}_{\Bbbk} \mathbb{Z}\mathbb{Z}^{\mathrm{op}})$.

5.3. **Comparison.** In this subsection, we investigate the relation among the distance d_{BL}, the convolution distance through the equivalence in Proposition 35, and ours.

Berkouk–Ginot–Oudot [3] considered a functor $\Xi\colon \operatorname{rep}_{\Bbbk} \mathbb{U} \to \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})^{\mathrm{op}}$ and $\Psi\colon \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})^{\mathrm{op}} \to \operatorname{rep}_{\Bbbk} \mathbb{U}$, where $\mathbb{U} = \{(a,b) \in \mathbb{R}^2 \mid b \geq a\}$. Moreover, they proved that the functors commute with $[\vec{\varepsilon}]$ and $(\text{-}) \star K_\varepsilon$ in the derived setting, where $\vec{\varepsilon} = (-\varepsilon, \varepsilon)$. Here we restate the result in a weaker form, i.e., in the non-derived setting. For $F \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$, we define

$$F \star_{\mathrm{nd}} K_\varepsilon := H^0(F \star K_\varepsilon) \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}}), \tag{5.14}$$

$$\phi^{\varepsilon}_{F,\mathrm{nd}} := H^0(\phi^{\varepsilon}_{F})\colon F \star_{\mathrm{nd}} K_\varepsilon \to F \text{ in } \operatorname{Sh}_c(\Bbbk_{\mathbb{R}}). \tag{5.15}$$

By Proposition 3.22, Lemmas 3.27 and 3.28 in [3] and Proposition 3.8 in [2], for $M \in \operatorname{rep}_{\Bbbk} \mathbb{U}$ whose indecomposable summands are only blocks of type $\mathbf{bb}^-, \mathbf{db}^+, \mathbf{hb}$, and $\mathbf{vb}$, one has $\Xi(M[\vec{\varepsilon}]) \cong M \star_{\mathrm{nd}} K_\varepsilon$ for any $\varepsilon \geq 0$. Similarly, for $F \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$ one has $\Psi(F)[\vec{\varepsilon}] \cong \Psi(F \star_{\mathrm{nd}} K_\varepsilon)$ for any $\varepsilon \geq 0$ (cf. Proposition 4.16 in [3]). Moreover, they satisfy $\Xi \circ \Psi \simeq \mathbb{1}$.

Now we consider the relation to the equivalence in Proposition 35. We define Θ as the composite $\operatorname{rep}_{\Bbbk} \mathbb{ZZ} \xrightarrow{D} (\operatorname{rep}_{\Bbbk} \mathbb{ZZ}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{S^{\mathrm{op}}} \operatorname{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})^{\mathrm{op}}$, where D denotes the $\Bbbk$-dual functor.

Proposition 36. *One has the following commutative diagram:*

$$\begin{tikzcd} \operatorname{rep}_{\Bbbk} \mathbb{ZZ} \arrow[r, "J"] \arrow[rd, "\Theta"'] & \operatorname{rep}_{\Bbbk} \mathbb{U} \arrow[d, "\Xi"] \\ & \operatorname{Sh}_{\mathbb{Z}}(\Bbbk_{\mathbb{R}})^{\mathrm{op}} \end{tikzcd} . \tag{5.16}$$

We define a non-derived version of the convolution distance as follows. For $F, G \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$, F and G are said to be *H^0-ε-isomorphic* if there exist $f\colon K_\varepsilon \star_{\mathrm{nd}} F \to G$ and $g\colon K_\varepsilon \star_{\mathrm{nd}} G \to F$ such that the following diagrams commute:

$$\begin{tikzcd} F^\bullet \star_{\mathrm{nd}} K_{2\varepsilon} \arrow[rr, "\phi^{2\varepsilon}_{F^\bullet}"] \arrow[rd, "f \star_{\mathrm{nd}} K_\varepsilon"'] & & F^\bullet \\ & G^\bullet \star_{\mathrm{nd}} K_\varepsilon \arrow[ru, "g"'] & \end{tikzcd} , \quad \begin{tikzcd} G^\bullet \star_{\mathrm{nd}} K_{2\varepsilon} \arrow[rr, "\phi^{2\varepsilon}_{G^\bullet}"] \arrow[rd, "g \star_{\mathrm{nd}} K_\varepsilon"'] & & G^\bullet \\ & F^\bullet \star_{\mathrm{nd}} K_\varepsilon \arrow[ru, "f"'] & \end{tikzcd} . \tag{5.17}$$

We define a non-derived convolution distance by

$$d_{C,\mathrm{nd}}(F,G) := \inf\{\varepsilon \geq 0 \mid F \text{ and } G \text{ are } H^0\text{-}\varepsilon\text{-isomorphic}\} \tag{5.18}$$

for $F, G \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$. It can be easily checked that for $F, G \in \operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$, the inequality

$$d_{C,\mathrm{nd}}(F,G) \leq d_C(F,G). \tag{5.19}$$

holds. In particular, if both F and G have no indecomposable direct summand of the form $\Bbbk_{(a,b)}$ with open interval (a,b) in $\mathbb{R}$, then the equality

$$d_{C,\mathrm{nd}}(F,G) = d_C(F,G) \tag{5.20}$$

holds (see Propositions 4.1 and 4.2 in [2]).

Through the functor Ξ, the interleaving distance $d_I^{\mathbb{U}}$ on $\operatorname{rep}_{\Bbbk} \mathbb{U}$ is compatible with $d_{C,\mathrm{nd}}$ on $\operatorname{Sh}_c(\Bbbk_{\mathbb{R}})$ as follows. This is a non-derived version of a main result of [3].

Lemma 37 (cf. Theorem 4.21 in [3])**.** *For any $M, N \in \mathrm{rep}_{\Bbbk}\,\mathbb{U}$ whose indecomposable summands are only blocks of type* $\mathbf{bb}^-, \mathbf{db}^+, \mathbf{hb}$, *and* $\mathbf{vb}$, *one has*

$$d_I^{\mathbb{U}}(M,N) = d_{C,nd}(\Xi(M),\Xi(N)). \tag{5.21}$$

The following shows that the distance d_{BL} coincides with the convolution distance on sheaves through the equivalence Θ.

Proposition 38. *For any $X, Y \in \mathrm{rep}_{\Bbbk}\,\mathbb{ZZ}$, one has*

$$d_{BL}(X,Y) = d_{C,nd}(\Theta(X),\Theta(Y)). \tag{5.22}$$

As a consequence of Proposition 38, our induced distance is incomparable with the non-derived and the ordinary convolution distance defined above in the purely zigzag setting. Indeed, for $X, Y \in \mathcal{Y}_c$, we have

$$d^{z_1}(X,Y) \le d_{BL}(X,Y) = d_{C,\mathrm{nd}}(\Theta(X),\Theta(Y)) = d_C(\Theta(X),\Theta(Y)). \tag{5.23}$$

On the other hand, for $X, Y \in \mathcal{Y}_{co}$, we have

$$d^{z_1}(X,Y) \ge d_{BL}(X,Y) = d_{C,\mathrm{nd}}(\Theta(X),\Theta(Y)) = d_C(\Theta(X),\Theta(Y)). \tag{5.24}$$

Note that by Corollary 29, $d^{z_1} := d_I^{E,\mathcal{A}} = d_B^{E,\mathcal{A}}$, where $\mathcal{A} = \mathrm{rep}_{\Bbbk}\,A_n(z_1)$ and $E^{-1} = \mathrm{RHom}(T,\text{-})$ with a fixed classical tilting module T and that $\mathcal{Y}_c$ and $\mathcal{Y}_{co}$ are the set of interval representations in $\mathrm{rep}_{\Bbbk}\,A_n(z_1)$ corresponds to closed and right-open interval representations of $\mathbb{ZZ}$. For the details of calculation, refer to Section 7 and Section 8 in [15].

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