ALGEBRAS ASSOCIATED TO NONCOMMUTATIVE CONICS IN QUANTUM PROJECTIVE PLANES

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ABSTRACT. The classification of noncommutative quadric hypersurfaces in quantum $\mathbb{P}^{n-1}$'s is a big project in noncommutative algebraic geometry and it is far away from complete. In this note, we mainly give some result about noncommutative conics in quantum $\mathbb{P}^2$'s.

1. INTRODUCTION

In this note, k is an algebraically closed filed of characteristic 0, all algebras and vector spaces are over k.

In (commutative) algebraic geometry, it is important to study the homogeneous coordinate ring $k[x_1, \cdots, x_n]/(f)$ of a quadric hypersurface in the projective space $\mathbb{P}^{n-1}$ where $0 \neq f \in k[x_1, \cdots, x_n]_2$. In noncommutative algebraic geomtry, we say the quotient algebra $S/(f)$ the *noncommutative quadric hypersurface* (*noncommutative conic* if $d = 3$) where S is a d-dimensional quantum polynomial algebra defined below and $0 \neq f \in S_2$ a regular central element.

Definition 1. A noetherian connected graded algebra S generated in degree 1 is called a *d-dimensional quantum polynomial algebra* if

(1) $\operatorname{gldim} S = d$,

(2) $\operatorname{Ext}^i_S(k, S(-j)) \cong \begin{cases} k & \text{if } i = j = d, \\ 0 & \text{otherwise,} \end{cases}$ and

(3) $H_S(t) := \sum_{i=0}^{\infty} (\dim_k S_i) t^i = 1/(1-t)^d$.

The classification of noncommutative quadric hypersurfaces is a big project in noncommutative algebraic geometry and it is far away from complete. The good thing is there are many notable developments in the study of noncommutative quadric hypersurfaces: Smith and Van den Bergh introduce a finite dimensional algebra $C(A)$ associated to $A := S/(f)$ which determines the Cohen-Macaulay representation of A (cf. [12]); Mori and Ueyama introduce the noncommutative matrix factorization of f over S and they also proved the noncommutative Knörrer's periodicity theorem (cf. [11]). He and Ye introduce the Clifford deformation associated to the pair (S, f) which is a nonhomogeneous PBW deformation and they showed that A is a noncommutative isolated singularity if and only if $C(A)$ is semisimple (cf. [5]), etc.

However, there is still no complete classification of noncommutative conics even though it should be the easiest case. Using theoretical tools above, and the fact that defining relations of all 3-dimensional quantum polynomial algebras are given by Itaba and Matsuno

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(cf. [8]), it is time for us to begin to work on the classification of noncommutative conics and it would be a good step forward to classify noncommutative quadric hypersurfaces. Thus in this note, we focus on the noncommutative conics, and give some results.

2. Preliminaries

Let S be a n-dimensional quantum polynomial algebra, we call the noncommutative projective scheme $\mathrm{Proj}_{nc} S$ associated to S which induced by Artin and Zhang (cf. [2]) the *quantum* $\mathbb{P}^{n-1}$. We start by repeating the following definition.

Definition 2. A quotient algebra $A = S/(f)$ is called a *noncommutative quadric hypersurface* (resp. *noncommutative conic*) in a quantum projective space $\mathrm{Proj}_{nc} S$ if S is an n-dimensional (resp. $n = 3$) quantum polynomial algebra and $0 \neq f \in S_2$ a regular central element.

Let $A = S/(f)$ be a noncommutative quadric hypersruface, $A^!$ and $S^!$ be the quadratic duals of A and S respectively. Then there is a unique $f^! \in A^!_2$ such that $A^!/(f^!) = S^!$. Define

$$C(A) := A^![f^{!^{-1}}]_0.$$

Theorem 3. [12] (1) $\dim_k C(A) = 2^{n-1}$.
(2) $\underline{\mathrm{CM}}^{\mathbb{Z}} A \cong D^b(\mathrm{mod}\, C(A))$, *where* $\underline{\mathrm{CM}}^{\mathbb{Z}} A$ *the stable category of the category of maximal Cohen-Macaulay graded right* A*-modules, and* $D^b(\mathrm{mod}\, C(A))$ *the bounded derived category of the category of finitely generated right* $C(A)$*-modules.*

Let $A = S/(f)$ be a noncommutative quadric hypersurface. Let $\mathrm{grmod}\, A$ be the category of finitely generated right A-modules, and $\mathrm{tor}\, A$ the full subcategory of $\mathrm{grmod}\, A$ consisting of finite dimensional graded right A-modules. Denote by $\mathrm{qgr}\, A := \mathrm{grmod}\, A/\,\mathrm{tor}\, A$ the quotient category.

Definition 4. [13] A is called a *noncommutative isolated singularity* if the global dimension $\mathrm{gldim}\, \mathrm{qgr}\, A < \infty$.

If A is commutative, then the above definition is equivalent to say that $\mathrm{Proj}\, A$ is smooth.

Theorem 5. [5] *A is a noncommutative isolated singularity if and only if $C(A)$ is semisimple.*

If we want to classify all noncommutative quadric hypersurface $A = S/(f)$, there are two steps:

(1) Classify quantum polynomial algebras S.
(2) Find all nonzero regular central elements in S_2 for each S.

For noncommutative conics $A = S/(f)$. The first step above is completed. We will also give some facts about 3-dimensional quantum polynomial algebras.

Theorem 6. [1] *Every 3-dimensional quantum polynomial algebra S is a domain.*

Then we would like to introduce the definition of superpotentials. Let $V = \mathrm{Span}\{x, y, z\}$ be a vector space. Let $\omega \in V^{\otimes 3}$. Define a linear map $\varphi : V^{\otimes 3} \to V^{\otimes 3}, v_1 \otimes v_2 \otimes v_3 \mapsto v_2 \otimes v_3 \otimes v_1$. The map φ can be regarded as a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ in the symmetric group $\mathrm{Sym}(3)$ of degree 3.

Definition 7. Let $\omega \in V^{\otimes 3}$, then ω is called a

1. *twisted superpotential* if there exists $\sigma \in \mathrm{GL}(V)$ such that $(\sigma \otimes \mathrm{id} \otimes \mathrm{id})\varphi(\omega) = \omega$;
2. *superpotential* if $\varphi(\omega) = \omega$ (i.e. ω is invariant under permutation φ);
3. *symmetric superpotential* if it is invariant under all permutations in the symmetric group $\mathrm{Sym}(3)$.

Example 8. (1) $xyz + yzx + zxy - (xzy + zyx + yxz)$ is a superpotential.
(2) $xyz + yzx + zxy + xzy + zyx + yxz$ is a symmetric superpotential.
(3) $xyz + yzx + zxy + xzy + zyx + yxz + x^3 + y^3 + z^3$ is a symmetric superpotential.

For $\omega \in V^{\otimes 3}$, there are unique $\omega_1, \omega_2, \omega_3 \in V^{\otimes 2}$ such that we can write

$$\omega = x \otimes \omega_1 + y \otimes \omega_2 + z \otimes \omega_3.$$

Then $\mathcal{D}(\omega) := T(V)/(\omega_1, \omega_2, \omega_3)$ is a quadratic algebra.

Example 9. Let $\omega = xyz + yzx + zxy - (xzy + zyx + yxz)$, then

$$\mathcal{D}(\omega) = k\langle x, y, z\rangle/(yz - zy, zx - xz, xy - yx) = k[x, y, z].$$

Theorem 10. [3, 4, 10] *Every 3-dimensional quantum polynomial algebra S is isomorphic to an algebra $\mathcal{D}(\omega)$ for some unique twisted superpotential $\omega \in V^{\otimes 3}$.*

Definition 11. A twisted superpotential $\omega \in V^{\otimes 3}$ is called *regular* if $\mathcal{D}(\omega)$ is a 3-dimensional quantum polynomial algebra.

Example 12. (1) $\omega = xyz + yzx + zxy - (xzy + zyx + yxz)$ is a regular superpotential.
(2) $\omega = xyz + yzx + zxy + xzy + zyx + yxz$ is a symmetric regular superpotential.
(3) Non-example: Let $\omega = xyz + yzx + zxy + xzy + zyx + yxz + x^3 + y^3 + z^3$. Though ω is a symmetric superpotential, $\mathcal{D}(\omega)$ is not a 3-dimensional quantum polynomial algebra.

3. Main results

In this note, we are mainly interested about a noncommutative conic A such that its quadratic dual $A^!$ is commutative.

Theorem 13. [6] *Let $A = S/(f)$ be a noncommutative conic. Then $A^!$ is commutative if and only if $S = \mathcal{D}(\omega)$, where ω is a symmetric regular superpotential. Moreover, there are only 4 types of symmetric regular superpotentals:*

(1) $xyz + yzx + zxy + xzy + zyx + yxz$,
(2) $xyz + yzx + zxy + xzy + zyx + yxz + x^3$,
(3) $xyz + yzx + zxy + xzy + zyx + yxz + x^3 + y^3$,
(4) $xyz + yzx + zxy + xzy + zyx + yxz + \lambda(x^3 + y^3 + z^3)$ *where* $\lambda \in k$ *such that* $\lambda^3 \neq 0, 1, -8$.

Note that as mentioned in the begining of the Preliminaries, to classify noncommutative conics, we also need to calculate central elements of 3-dimensional quantum polynomial algebras. We write $\omega_1, \omega_2, \omega_3, \omega_4$ for above superpotentials respectively. Algebras $\mathcal{D}(\omega_1), \mathcal{D}(\omega_2), \mathcal{D}(\omega_3)$ have PBW bases, so their central elements are easy to calculate. However, algebras

$$\mathcal{D}(\omega_4) \cong S^{1,1,\lambda} := k\langle x, y, z\rangle/(yz + zy + \lambda x^2, xz + zx + \lambda y^2, xy + yz + \lambda x^2)$$

where $\lambda \in k$ and $\lambda^3 \neq 0, 1, -8$ are 3-dimensional Sklyanin algebras which do not have PBW bases (cf. [9]). It is not easy to calculate their center directly. Fortunately, we showed the following result.

Lemma 14. [6] *$Z(\mathcal{D}(\omega_i))_2 = kx^2 + ky^2 + kz^2$, where $i = 1, 2, 3, 4$.*

Thus for our case, a noncommutative conic A can be written as $\mathcal{D}(\omega_i)/(\alpha x^2+\beta y^2+\gamma z^2)$ where $i = 1, 2, 3, 4$ and $\alpha, \beta, \gamma \in k$. Since by Theorem 3, there is a 4-dimensional algebra $C(A)$ determines the Cohen-Macaulaly representation of A, we would like to give the classification of all $C(A)$.

Theorem 15. [6] *For noncommutative conics $A = S/(f)$ associated to symmetric regular superpotentials, the set of isomorphism classes of algebras $C(A)$ is equal to the set of isomorphism classes of 4-dimensional commutative Frobenius algebras. They are*

$$k^4, k[u]/(u^2) \times k^2, k[u]/(u^3) \times k, (k[u]/(u^2))^{\times 2}, k[u]/(u^4), k[u, v]/(u^2, v^2).$$

Corollary 16. *For two noncommutative conics A, A' in our case. $\underline{\mathrm{CM}}^{\mathbb{Z}} A \cong \underline{\mathrm{CM}}^{\mathbb{Z}} A'$ if and only if $C(A) \cong C(A')$.*

Then we consider Sklyanin algebras $S^{1,1,\lambda}$. Unlike the other 3 cases, there are infinitely isomorphism classes of $S^{1,1,\lambda}$. So the classification of $C(A)$ where $A = S^{1,1,\lambda}/(f)$ is the most hard case.

For $0 \neq f \in Z(S^{1,1,\lambda})_2$, denote by $K_f := \{g \in (S^{1,1,\lambda})_1$ up to sigh such that $g^2 = f\}$.

Theorem 17. [7] *Let $A = S^{1,1,\lambda}/(f), A' = S^{1,1,\lambda'}/(f')$ be two noncommutative conics in Sklyanin quantum $\mathbb{P}^2$'s. Then $C(A) \cong C(A')$ if and only if $\#(K_f) = \#(K_{f'})$. Moreover, exactly one of the following holds:*

(1) *$\#(K_f) = 2$ and $C(A) \cong k[u]/(u^3) \times k$.*
(2) *$\#(K_f) = 3$ and $C(A) \cong k[u]/(u^2) \times k^2$.*
(3) *$\#(K_f) = 4$ and $C(A) \cong k^4$.*

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