CHARACTERIZATION OF THE QUANTUM PROJECTIVE PLANES FINITE OVER THEIR CENTERS

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Abstract. For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, Artin-Tate-Van den Bergh showed that A is finite over its center if and only if $|\sigma| < \infty$. Moreover, Artin showed that if A is finite over its center and $E \neq \mathbb{P}^2$, then A has a fat point module, which plays an important role in noncommutative algebraic geometry, however, the converse is not true in general. In this paper, we show that, if $E \neq \mathbb{P}^2$, then A has a fat point module if and only if the quantum projective plane $\mathsf{Proj}_{\rm nc} A$ is finite over its center if and only if $|\nu^*\sigma^3| < \infty$ where ν is the Nakayama automorphism of A. As a byproduct, we show that $|\nu^*\sigma^3| = 1$ or ∞ if and only if the isomorphism classes of simple 2-regular modules over ∇A are parameterized by $E \subset \mathbb{P}^2$.

1. Geometric quantum polynomial algebras

Throughout this paper, let k be an algebraically closed field of characteristic 0. All graded algebras are finitely generated in degree 1 over k, that is, $A \cong k\langle x_1, \dots, x_n\rangle/I$, where I is a two-sided homogeneous ideal of $k\langle x_1, \dots, x_n\rangle$, $\deg x_i = 1$ $(\forall i = 1, \dots, n)$. We denote by $\mathsf{grmod}\, A$ the category of finitely generated graded right A-modules. The $(n-1)$-dimensional projective space over k is denoted by $\mathbb{P}^{n-1}_k (= \mathbb{P}^{n-1})$.

Definition 1 ([3]). A right noetherian graded algebra A is called a *d-dimensional quantum polynomial algebra if*

(i) $\operatorname{gldim} A = d$,

(ii) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (*Gorenstein condition*)

(iii) $H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1-t)^{-d}$ (*Hilbert series of A*).

For example, a polynomial algebra $k[x_1, \dots, x_d]$ is a commutative d-dimensional quantum polynomial algebra. Also, a skew polynomial algebra $k\langle x_1, \dots, x_d\rangle/(x_j x_i - \alpha_{i,j} x_i x_j)$, $(1 \leq i < j \leq d,\ \alpha_{i,j} \in k\backslash\{0\})$ is a d-dimensional quantum polynomial algebra. In [3], A is a 3-dimensional quantum polynomial algebra if and only if A is a 3-dimensional quadratic AS-regular algebra, which is of the form $A \cong k\langle x, y, z\rangle/(f_1, f_2, f_3)$, $f_i \in k\langle x, y, z\rangle_2$ $(i = 1, 2, 3)$.

Next, we recall a notion of geometric algebra for a quadratic algebra. Let E be a projective scheme in $\mathbb{P}^{n-1}$, and $\sigma \in \operatorname{Aut}_k E$. Here, we consider a quadratic algebra

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$A = k\langle x_1, \dots, x_n\rangle/I$ where I is a homogeneous ideal of $k\langle x_1, \dots, x_n\rangle$ generated by I_2. We set $\mathcal{V}(I_2) := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \,|\, f(p, q) = 0, \forall f \in I_2\}$.

Definition 2 ([14])**.** (1) We say that A *satisfies* (G1) if there exists a geometric pair (E, σ) such that

$$\mathcal{V}(I_2) = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \,|\, p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$, and call E the *point scheme* of A.

(2) We say that A *satisfies* (G2) if there exists a geometric pair (E, σ) such that

$$I_2 = \{f \in k\langle x_1, \dots, x_n\rangle_2 \,|\, f(p, \sigma(p)) = 0 \text{ for any } p \in E\}.$$

In this case, we write $A = \mathcal{A}(E, \sigma)$.

(3) A quadratic algebra A is called *geometric* if A satisfies both (G1) and (G2) with $A = \mathcal{A}(\mathcal{P}(A))$.

Suppose that E is a triangle in $\mathbb{P}^2$, and $\sigma \in \operatorname{Aut}_k E$ stabilizes each component. Then, $A := \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$ is a 3-dimensional geometric quantum polynomial algebra, where $\alpha, \beta, \gamma \in k$ such that $\alpha\beta\gamma \neq 0, 1$.

Theorem 3 ([5])**.** *Every 3-dimensional quantum polynomial algebra is geometric where the point scheme is either $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$.*

Note that the classification of 3-dimensional quantum polynomial algebras $A = \mathcal{A}(E, \sigma)$ reduces to the classification of geometric pairs (E, σ). A type of a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ is defined in terms of the point scheme $E \subset \mathbb{P}^2$ (for example, see [9, Subsection 2.3]):

Type P: E is $\mathbb{P}^2$.
Type S: E is a triangle.
Type S': E is a union of a line and a conic meeting at two points.
Type T: E is a union of three lines meeting at one point.
Type T': E is a union of a line and a conic meeting at one point.
Type NC: E is a nodal cubic curve.
Type CC: E is a cuspidal cubic curve.
Type TL: E is a triple line.
Type WL: E is a union of a double line and a line.
Type EC: E is an elliptic curve.

2. Quantum projective spaces finite over their centers

In this section, we recall a quantum projective space from Artin-Zhang [7]. For a right noetherian graded algebra A, $\mathsf{tors}\, A$ is the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules over k.

Definition 4 ([7])**.** (1) *The noncommutative projective scheme associated to* A is defined by $\mathsf{Proj}_{\rm nc} A = (\mathsf{tails}\, A, \pi A)$ where $\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A$ is the quotient category, $\pi : \mathsf{grmod}\, A \to \mathsf{tails}\, A$ is the quotient functor, and $A \in \mathsf{grmod}\, A$ is a regular module.

(2) Moreover, if A is a d-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\rm nc} A$ is called a *quantum* $\mathbb{P}^{d-1}$. In particular, for the case that $d = 3$, $\mathsf{Proj}_{\rm nc} A$ is called a *quantum projective plane.*

We remark that, if A is commutative, then $\mathsf{Proj}_{\mathrm{nc}}\, A \cong \mathsf{Proj}\, A$. It is known that if A is a 2-dimensional quantum polynomial algebra, then $\mathsf{Proj}_{\mathrm{nc}} A \cong \mathbb{P}^1$.

Now, we mention the relationships between a 3-dimensional quantum polynomial algebra and a quantum projective plane.

Theorem 5 ([2])**.** *Let A and A' be 3-dimensional quantum polynomial algebras. Then*

$$\mathsf{grmod}\, A \cong \mathsf{grmod}\, A' \text{ if and only if } \mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A'.$$

Lemma 6 ([10])**.** *For every 3-dimensional quantum polynomial algebra A, there exists a 3-dimensional Calabi-Yau quantum polynomial algebra A' such that $\mathsf{grmod}\, A \cong \mathsf{grmod}\, A'$ so that $\mathsf{Proj}_{\mathrm{nc}}\, A \cong \mathsf{Proj}_{\mathrm{nc}}\, A'$.*

Here, a quantum polynomial algebra A' is called *Calabi-Yau* if the Nakayama automorphism of A' is the identity. The above results play an essential role to prove our main results. Note that, Lemma 6 claims that every quantum projective plane has a 3-dimensional Calabi-Yau quantum polynomial algebra as a homogeneous coordinate ring.

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, we have the following geometric characterization when A is finite over its center.

Theorem 7 ([6])**.** *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quantum polynomial algebra, then $|\sigma| < \infty$ if and only if A is finite over its center.*

To prove Theorem 7, *fat points of a quantum projective plane* $\mathsf{Proj}_{\mathrm{nc}}\, A$ play an essential role. By Artin [1], if A is finite over its center and $E \neq \mathbb{P}^2$, then $\mathsf{Proj}_{\mathrm{nc}}\, A$ has a *fat point*, however, the converse is not true.

Definition 8. Let A be a graded algebra. *A point of* $\mathsf{Proj}_{\mathrm{nc}} A$ is an isomorphism class of a simple objects of the form $\pi M \in \mathsf{tails}\, A$ where $M \in \mathsf{grmod}\, A$ is a graded right A-module such that $\lim_{i\to\infty} \dim_k M_i < \infty$. A point πM is called *fat* if $\lim_{i\to\infty} \dim_k M_i > 1$. In this case M is called *a fat point module over A*.

To characterize "geometric" quantum projective spaces finite over their centers, the following notion was introduced:

Definition 9 ([15])**.** For a geometric pair (E, σ) where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$, we define

$$\mathrm{Aut}_k(\mathbb{P}^{n-1}, E) := \{\phi|_E \in \mathrm{Aut}_k E \mid \phi \in \mathrm{Aut}_k \mathbb{P}^{n-1}\},$$

and $\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i \in \mathrm{Aut}_k(\mathbb{P}^{n-1}, E)\}$, which is called *the norm of σ*.

For a geometric pair (E, σ), clearly $\|\sigma\| \leq |\sigma|$ holds. The following facts will be used to prove our main results.

Lemma 10 ([1], [15])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra.*
(1) $\|\sigma\| = 1 \iff E = \mathbb{P}^2$.
(2) $1 < \|\sigma\| < \infty \iff \mathsf{Proj}_{\mathrm{nc}} A$ *has a fat point.*

Definition 11 ([15], ([11]))**.** Let A be a d-dimensional quantum polynomial algebra. We say that $\mathsf{Proj}_{\mathrm{nc}} A$ is *finite over its center* if there exists a d-dimensional quantum polynomial algebra A' finite over its center such that

$$\mathsf{grmod}\, A \cong \mathsf{grmod}\, A' \ (\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A').$$

Theorem 12 ([15])**.** *Suppose that $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quantum polynomial algebra where E is a triangle in $\mathbb{P}^2$ (that is, A is of Type S). Then $\|\sigma\| < \infty$ if and only if $\mathsf{Proj}_{\rm nc} A$ is finite over its center.*

The aim of this paper is to extend Theorem 12 to all types.

3. Main results

In this section, we give our main results of this paper. First, we describe the result for a 3-dimensional Calabi-Yau quantum polynomial algebra.

Theorem 13 ([11])**.** *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra, then $||\sigma|| = |\sigma^3|$, so the following are equivalent*:
(1) $|\sigma| < \infty$.
(2) $||\sigma|| < \infty$.
(3) *A is finite over its center.*
(4) *$\mathsf{Proj}_{\rm nc} A$ is finite over its center.*

For a 3-dimensional quantum polynomial algebra, we need the following definition and lemma by Mori-Ueyama [16]:

Definition 14 ([16])**.** For a d-dimensional geometric quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ with the Nakayama automorphism $\nu \in \operatorname{Aut} A$, we define a graded algebra $\overline{A} := \mathcal{A}(E, \nu^* \sigma^d)$ satisfying (G2) in Definition 2.

Lemma 15 ([16])**.** *If A and A' are geometric quantum polynomial algebras, then*

$$\mathsf{grmod}\, A \cong \mathsf{grmod}\, A' \implies \overline{A} \cong \overline{A'}.$$

Using Lemma 6, Lemma 15, Theorem 13 and other results, we have the following theorem.

Theorem 16 ([11])**.** *If $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quantum polynomial algebra with the Nakayama automorphism $\nu \in \operatorname{Aut} A$, then $||\sigma|| = |\nu^* \sigma^3|$, so the following are equivalent*:
(1) $|\nu^* \sigma^3| < \infty$.
(2) $||\sigma|| < \infty$.
(3) *$\mathsf{Proj}_{\rm nc} A$ is finite over its center.*
Moreover, if A is of Type T, T', CC, TL, WL, then A is never finite over its center.

As a corollary, we have the following.

Corollary 17 ([11])**.** *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra such that $E \neq \mathbb{P}^2$, and $\nu \in \operatorname{Aut} A$ the Nakayama automorphism of A. Then the following are equivalent*:
(1) $|\nu^* \sigma^3| < \infty$.
(2) $\|\sigma\| < \infty$.
(3) *$\mathsf{Proj}_{\rm nc} A$ is finite over its center.*
(4) *$\mathsf{Proj}_{\rm nc} A$ has a fat point.*

Example 18. Let

$$A = \mathcal{A}(E,\sigma) = k\langle x,y,z\rangle/(yz-\alpha zy, zx-\beta xz, xy-\gamma yx),\ (\alpha,\beta,\gamma \in k, \alpha\beta\gamma \neq 0,1)$$

be a 3-dimensional quantum polynomial algebra of Type S, where $E = \mathcal{V}(x)\cup\mathcal{V}(y)\cup\mathcal{V}(z) \subset \mathbb{P}^2$ and $\sigma \in \operatorname{Aut}_k E$ is given by $\begin{cases} \sigma(0,b,c) = (0,b,\alpha c), \\ \sigma(a,0,c) = (\beta a,0,c), \\ \sigma(a,b,0) = (a,\gamma b,0). \end{cases}$ By calculating the Nakayama automorphism ν of A, we have $\nu^* = \begin{pmatrix} \gamma/\beta & 0 & 0 \\ 0 & \alpha/\gamma & 0 \\ 0 & 0 & \beta/\alpha \end{pmatrix}$, so $\begin{cases} \nu^*\sigma^3(0,b,c) = (0,b,\alpha\beta\gamma c), \\ \nu^*\sigma^3(a,0,c) = (\alpha\beta\gamma a,0,c), \\ \nu^*\sigma^3(a,b,0) = (a,\alpha\beta\gamma b,0). \end{cases}$

(1) By Theorem 7, $|\sigma| = \operatorname{lcm}(|\alpha|,|\beta|,|\gamma|) < \infty \iff A$ is finite over its center.
(2) By Collorary 17, $\|\sigma\| = |\nu^*\sigma^3| = |\alpha\beta\gamma| < \infty \iff \mathsf{Proj}_{\rm nc}A$ is finite over its center $\iff$ $\mathsf{Proj}_{\rm nc}A$ has a fat point.

Finally, we apply our results above to representation theory of finite dimensional algebras.

Definition 19 ([8])**.** Let R be a finite dimensional algebra of $\operatorname{gldim} R = d < \infty$. We define an autoequivalence $\nu_d \in \operatorname{Aut} D^b(\mathsf{mod} R)$ by $\nu_d(M) := M \otimes^{\rm L}_R DR[-d]$ where $D^b(\mathsf{mod} R)$ is the bounded derived category of $\mathsf{mod} R$ and $DR := \operatorname{Hom}_k(R,k)$. We say that R is *d-representation infinite* if $\nu_d^{-i}(R) \in \mathsf{mod} R$ for all $i \in \mathbb{N}$. In this case, we say that a module $M \in \mathsf{mod} R$ is *d-regular* if $\nu_d^i(M) \in \mathsf{mod} R$ for all $i \in \mathbb{Z}$.

By Minamoto [12], a 1-representation infinite algebra is exactly the same as a finite dimensional hereditary algebra of infinite representation type. For representation theory of such an algebra, regular modules play an essential role.

For a d-dimensional quantum polynomial algebra A, *the Beilinson algebra of A* is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ 0 & A_0 & \cdots & A_{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

We remark that the Beilinson algebra is a typical example of a $(d-1)$-representation infinite algebra by Minamoto-Mori [13, Theorem 4.12]. To investigate representation theory of such an algebra, it is important to classify simple $(d-1)$-regular modules.

Corollary 20 ([11])**.** *Let $A = \mathcal{A}(E,\sigma)$ be a 3-dimensional quantum polynomial algebra with the Nakayama automorphism $\nu \in \operatorname{Aut} A$. Then the following are equivalent:*

(1) $|\nu^*\sigma^3| = 1$ *or* ∞.
(2) $\mathsf{Proj}_{\rm nc}A$ *has no fat point.*
(3) *The isomorphism classes of simple* 2*-regular modules over* ∇A *are parameterized by the set of closed points of* $E \subset \mathbb{P}^2$.

In particular, if A is of Type P, T, T', CC, TL, WL, then A satisfies all of the above conditions.

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