# HOCHSCHILD COHOMOLOGY OF $\mathrm{N}_m$

TOMOHIRO ITAGAKI, KAZUNORI NAKAMOTO, AND TAKESHI TORII

Abstract. Let $\mathrm{N}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{11} = a_{22} = \cdots = a_{mm}$ and $a_{ij} = 0$ for any $i > j\}$ for a commutative ring $R$. We calculate the Hochschild cohomology ring $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ as $R$-algebras. We also calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ as $R$-modules.

*Key Words*: Hochschild cohomology, Koszul algebra, Spectral sequence.

2010 *Mathematics Subject Classification*: Primary 16E40; Secondary 16S37, 18G40.

## 1. Introduction

Let $R$ be a commutative ring. For $m \geq 3$, set

$$\mathrm{N}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{11} = a_{22} = \cdots = a_{mm} \text{ and } a_{ij} = 0 \text{ for any } i > j\}.$$

Setting $x_1 = E_{1,2}, x_2 = E_{2,3}, \ldots, x_{m-1} = E_{m-1,m}$, we have an isomorphism as $R$-algebras:

$$\mathrm{N}_m(R) \cong R\langle x_1, x_2, \ldots, x_{m-1}\rangle / \langle x_i x_j \mid j \neq i+1\rangle,$$

where $E_{i,j} \in \mathrm{M}_m(R)$ denotes the matrix with entry 1 in the $(i,j)$-component and 0 the other components. Since $\mathrm{N}_m(R)$ is a quadratic monomial algebra over $R$, it is Koszul. The Koszul dual $\mathrm{N}_m(R)^!$ of $\mathrm{N}_m(R)$ is isomorphic to $R\langle y_1, y_2, \ldots, y_{m-1}\rangle / \langle y_i y_{i+1} \mid 1 \leq i \leq m-2\rangle$. Put

$$\varphi(d) = \mathrm{rank}_R \mathrm{N}_m(R)^!_d,$$

where $|y_i| = 1$ and $\mathrm{N}_m(R)^!_d$ is the homogeneous part of $\mathrm{N}_m(R)^!$ of degree $d$. The Poincaré series $f^!(t) = \sum_{d \geq 0} \varphi(d) t^d$ can be calculated by

$$f^!(t) = \frac{1}{1 + \sum_{k=1}^{m-1} (-1)^k (m-k) t^k}.$$

In this paper, we calculate the Hochschild cohomology $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ as $R$-modules. We also calculate $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ as $R$-algebras. In the previous talk "An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring II", we reported that we calculated the Hochschild cohomology $\mathrm{HH}^*(A, \mathrm{M}_3(k)/A)$ for any 26 types of $k$-subalgebras $A$ of $\mathrm{M}_3(k)$ over an algebraically closed filed $k$. Then $\mathrm{N}_3(k)$ is one of the most difficult $k$-subalgebras $A$ of $\mathrm{M}_3(k)$ to calculate $\mathrm{HH}^*(A, \mathrm{M}_3(k)/A)$ (for

The detailed version of this paper will be submitted for publication elsewhere.

The first author was partially supported by JSPS KAKENHI Grant Number JP17K14175. The second author was partially supported by JSPS KAKENHI Grant Number JP20K03509. The third author was partially supported by JSPS KAKENHI Grant Numbers JP17K05253.

details, see [4]). This time, we would like to calculate the case $\mathrm{N}_m(R)$ for $m \geq 3$. As a result of our calculation, we will obtain how to calculate Hochschild cohomology for some type of algebras $A$.

The main theorems of this paper are the following:

**Theorem 1.** *Let $m \geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ is a free $R$-module for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ for $n \geq 0$ is given by*

$$\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = \begin{cases} m-1 & (n=0) \\ (m-2)\varphi(n) & (n>0). \end{cases}$$

**Theorem 2.** *Let $m \geq 3$. The Hochschild cohomology $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is a free $R$-module for $n \geq 0$. The rank of $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is given by*

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R)) \\ = &\begin{cases} 2 & (n=0) \\ 2m-4 & (n=1) \\ \varphi(n) + (m-4)\varphi(n-1) & \\ \quad +(-1)^m \varphi(n-m+1) + \displaystyle\sum_{k=2}^{m-1} (-1)^k (k+1)\varphi(n-k) & (n \geq 2). \end{cases} \end{aligned}$$

**Theorem 3.** *Let $m \geq 3$. There is an augmentation map $\epsilon : \mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R)) \to R$ as an $R$-algebra homomorphism such that the Kernel $\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ of $\epsilon$ satisfies*

$$\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \cdot \overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) = 0.$$

*In particular, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is an infinitely generated algebra over $R$.*

## 2. Preliminaries

In this section, we make a brief survey of Hochschild cohomology (cf. [1] and [6]). We also explain four steps for proving the main theorems.

**Definition 4** (Hochschild cohomology)**.** Let $A$ be an associative algebra over a commutative ring $R$. Let $M$ be an $A$-bimodule. Assume that $A$ is projective over $R$. Let $A^e := A \otimes_R A^{op}$ be the enveloping algebra of $A$. We regard $M$ as a left $A^e$-module. We define the $i$-th *Hochschild cohomology group* $H^i(A, M)$ as $\mathrm{Ext}^i_{A^e}(A, M)$.

**Proposition 5.** *Let $R$, $A$, and $M$ be as above. We can calculate $H^i(A, M)$ by taking the cohomology groups of the bar complex $(C^i(A, M), d^i)_{i \in \mathbb{Z}}$ which is given by*

$$C^i(A, M) := \begin{cases} \mathrm{Hom}_R(A^{\otimes i}, M) & (i \geq 0) \\ 0 & (i < 0) \end{cases}$$

*and* $d^i : C^i(A,M) \to C^{i+1}(A,M)$ $(i \geq 0)$ *defined by*

$$d^i(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_{i+1})$$
$$:= a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) + (-1)^{i+1} f(a_1 \otimes a_2 \otimes \cdots \otimes a_i) a_{i+1}$$

*for* $f \in C^i(A,M)$ $(i \geq 1)$ *and*

$$d^0(m)(a) = am - ma$$

*for* $m \in C^0(A,M) = M$. *Here the tensor products are over* $R$.

Let $N$ be another $A$-bimodule over $R$. We define a map

$$\cup : C^*(A,M) \times C^*(A,N) \longrightarrow C^*(A, M \otimes_A N)$$

by

$$(f \cup g)(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q) = f(a_1 \otimes \cdots \otimes a_p) \otimes g(b_1 \otimes \cdots \otimes b_q)$$

for $f \in C^p(A,M)$ and $g \in C^q(A,N)$. The map $\cup$ is $R$-bilinear and satisfies

$$d^{p+q}(f \cup g) = d^p(f) \cup g + (-1)^p f \cup d^q(g).$$

Hence the map $\cup$ induces a map

$$\mathrm{HH}^p(A,M) \otimes_R \mathrm{HH}^q(A,N) \longrightarrow \mathrm{HH}^{p+q}(A, M \otimes_A N)$$

of $R$-modules. In particular, $\mathrm{HH}^*(A,A)$ becomes a graded associative algebra over $R$ by the cup product $\cup$.

We divide the proof of the main theorems into four steps.

(Step 1) Show that $\mathrm{HH}^*(\mathrm{N}_m(R), R) \cong \mathrm{N}_m(R)^!$ as graded algebras over $R$.

(Step 2) For a $\mathbb{Z}$-graded $\mathrm{N}_m(R)$-bimodule $M = \mathrm{N}_m(R)$ or $M = \mathrm{M}_m(R)/\mathrm{N}_m(R)$, consider a filtration of $\mathbb{Z}$-graded $\mathrm{N}_m(R)$-bimodules over $R$:

$$M = F^{-(m-1)}M \supset F^{-(m-2)}M \supset \cdots \supset F^m M = 0.$$

Set $\mathrm{Gr}^p(M) = F^p M / F^{p+1} M$. Construct a spectral sequence

$$E_1^{p,q} \cong \mathrm{HH}^{p+q}(\mathrm{N}_m(R), \mathrm{Gr}^p(M)) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}_m(R), M),$$

which collapses from the $E_2$-page.

(Step 3) Calculate $E_2^{p,q}$.

(Step 4) Determine the product structure on $E_\infty^{p,q}$ for $M = \mathrm{N}_m(R)$.

In the following sections, we discuss four steps.

## 3. Step 1

Let

$$\mathrm{J} = \left\{ \begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{N}_m(R) \right\} \subset \mathrm{N}_m(R).$$

We calculate $\mathrm{HH}^*(\mathrm{N}_m(R), R)$ for the $\mathrm{N}_m(R)$-bimodule $\mathrm{N}_m(R)/\mathrm{J} \cong R$ over $R$. In this section, we show that $\mathrm{HH}^*(\mathrm{N}_m(R), R) \cong \mathrm{N}_m(R)^!$ as graded algebras over $R$.

Recall that the Koszul dual $\mathrm{N}_m(R)^!$ is isomorphic to $R\langle y_1, y_2, \dots, y_{m-1}\rangle/\langle y_i y_{i+1} \mid 1 \leq i \leq m-2\rangle$ as an $R$-algebra, Setting $|y_i| = 1$ for $1 \leq i \leq m-1$, $\mathrm{N}_m(R)^!$ can be regarded as a graded algebra over $R$. Let $\mathrm{N}_m(R)^!_d$ be the homogeneous part of $\mathrm{N}_m(R)^!$ of degree $d$. We denote by $\mathcal{B}(\mathrm{N}^!_d)$ the $R$-basis of $\mathrm{N}_m(R)^!_d$ consisting of monomials of degree $d$. Note that $\mathcal{B}(\mathrm{N}^!_0) = \{1\}$.

Set $\mathrm{N} = \mathrm{N}_m(R)$ and $\mathrm{N}^!_d = \mathrm{N}_m(R)^!_d$. By [5, Theorem 3], there is a projective resolution $\mathbb{P}$ of $\mathrm{N}_m(R)$ as $\mathrm{N}_m(R)$-bimodules over $R$ which is given by $P_n = \mathrm{N} \otimes_R \mathrm{N}^!_n \otimes_R \mathrm{N} = \oplus_{p \in \mathcal{B}(\mathrm{N}^!_n)} \mathrm{N} \otimes_R Rp \otimes_R \mathrm{N}$,

$$\cdots \to \mathrm{N} \otimes_R \mathrm{N}^!_2 \otimes_R \mathrm{N} \xrightarrow{d_1} \mathrm{N} \otimes_R \mathrm{N}^!_1 \otimes_R \mathrm{N} \xrightarrow{d_0} \mathrm{N} \otimes_R \mathrm{N} \xrightarrow{\mu} \mathrm{N} \to 0, \tag{3.1}$$

$\mu(a \otimes b) = ab$, and $d_{n-1}(1 \otimes p_n \otimes 1) = x_{j'} \otimes p^R_n \otimes 1 + (-1)^n 1 \otimes p^L_n \otimes x_j$, where $p_n = p^L_n y_j = y_{j'} p^R_n \in \mathrm{N}^!_n$ and $p^L_n, p^R_n \in \mathrm{N}^!_{n-1}$.

**Proposition 6.** $\mathrm{HH}^*(\mathrm{N}_m(R), R) \cong \mathrm{N}_m(R)^!$ *as graded algebras over* $R$.

*Proof.* By taking $\mathrm{Hom}_{\mathrm{N}^e}(-, R)$ of $\mathbb{P}$ in (3.1), we have

$$0 \to \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N} \otimes_R \mathrm{N}, R) \xrightarrow{\delta^0} \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N} \otimes_R \mathrm{N}^!_1 \otimes_R \mathrm{N}, R) \xrightarrow{\delta^1} \mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N} \otimes_R \mathrm{N}^!_2 \otimes_R \mathrm{N}, R) \to \cdots.$$

Since $\mathrm{J}R = R\mathrm{J} = 0$, $\delta^i = 0$ for $i \geq 0$. Hence, for each $i \geq 0$, $\mathrm{HH}^i(\mathrm{N}_m(R), R) \cong H^i(\mathrm{Hom}_{\mathrm{N}^e}(\mathrm{N} \otimes_R \mathrm{N}^!_* \otimes_R \mathrm{N}, R)) \cong \mathrm{N}^!_i$. We can also prove that $\mathrm{HH}^*(\mathrm{N}_m(R), R)$ is isomorphic to $\mathrm{N}_m(R)^!$ as graded algebras over $R$. □

## 4. Step 2

In this section, we construct a spectral sequence

$$E^{p,q}_1 \cong \mathrm{HH}^{p+q}(\mathrm{N}_m(R), \mathrm{Gr}^p(M)) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}_m(R), M)$$

for the $\mathbb{Z}$-graded $\mathrm{N}_m(R)$-bimodules $M = \mathrm{N}_m(R)$ or $\mathrm{M}_m(R)/\mathrm{N}_m(R)$, which collapses from the $E_2$-page.

We can choose an $R$-basis $\{E_{i,j} \mid 1 \leq i, j \leq m\}$ of $\mathrm{M}_m(R)$. Set

$$\mathrm{M}_r = \bigoplus_{j-i=r} R\{E_{i,j}\}.$$

Then $\mathrm{M}_m(R) = \oplus_{r\in\mathbb{Z}}\mathrm{M}_r$ is a $\mathbb{Z}$-graded associative algebra over $R$. Note that $\mathrm{N}_m(R)$ is a $\mathbb{Z}$-graded subalgebra of $\mathrm{M}_m(R)$. We also see that $\mathrm{M}_m(R)/\mathrm{N}_m(R)$ is a $\mathbb{Z}$-graded $\mathrm{N}_m(R)$-bimodule.

Let $M = \mathrm{N}_m(R)$ or $M = \mathrm{M}_m(R)/\mathrm{N}_m(R)$. Recall $\mathrm{J} = \oplus_{r>0}M_r$ in Step 1. For $\mathrm{N}_m(R)$-bimodule $M$ over $R$, set

$$F^pM = \sum_{a+b=p+(m-1)} \mathrm{J}^a M \mathrm{J}^b.$$

Then we have a filtration of $\mathrm{N}_m(R)$-bimodules

$$M = F^{-(m-1)}M \supset F^{-(m-2)}M \supset \cdots \supset F^{m-1}M \supset F^mM = 0$$

over $R$.

Let $\{C^*(\mathrm{N}_m(R), M)\}$ be the bar complex with coefficients in $M$. From the filtration $\{F^pM\}$, we obtain a filtration $\{C^*(\mathrm{N}_m(R), F^pM)\}$ on $C^*(\mathrm{N}_m(R), M)$. By a standard discussion, we obtain a spectral sequence (for details, see [3, Theorem 2.6]).

**Theorem 7.** *There is a spectral sequence of $R$-modules*

$$E_1^{p,q}(\mathrm{N}_m(R), M) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}_m(R), M),$$

*where*

$$E_1^{p,q}(\mathrm{N}_m(R), M) \cong \mathrm{HH}^{p+q}(\mathrm{N}_m(R), F^pM/F^{p+1}M).$$

For a $\mathbb{Z}$-graded $\mathrm{N}_m(R)$-bimodule $M = \mathrm{N}_m(R)$ or $\mathrm{M}_m(R)/\mathrm{N}_m(R)$ over $R$, we can define a $\mathbb{Z}$-grading on $C^p(\mathrm{N}_m(R), M)$ by the isomorphism

$$\begin{aligned} C^p(\mathrm{N}_m(R), M) &\cong \mathrm{Hom}_R(\mathrm{N}_m(R)^{\otimes p}, M) \\ &\cong (\mathrm{N}_m(R)^*)^{\otimes p} \otimes_R M, \end{aligned}$$

where $\mathrm{N}_m(R)^* = \mathrm{Hom}_R(\mathrm{N}_m(R), R)$. We denote by $C^{p,s}(\mathrm{N}_m(R), M)$ the degree $s$ part of $C^p(\mathrm{N}_m(R), M)$. For example,

$$E_{1,3} \in C^{0,-2}(\mathrm{N}_m(R), M),\ E_{1,2}^* \otimes E_{1,4}^* \otimes E_{1,2} \in C^{2,3}(\mathrm{N}_m(R), M),$$

where $\{I_m^*\} \cup \{E_{i,j}^* \mid i < j\}$ is the dual basis of $\mathrm{N}_m(R)^*$ with respect to the $R$-basis $\{I_m\} \cup \{E_{i,j} \mid i < j\}$ of $\mathrm{N}_m(R)$.

The differential $d : C^p(\mathrm{N}_m(R), M) \to C^{p+1}(\mathrm{N}_m(R), M)$ preserves the $\mathbb{Z}$-grading. Hence, $\{C^{*,s}(\mathrm{N}_m(R), M)\}$ becomes a subcomplex of $\{C^*(\mathrm{N}_m(R), M)\}$. We set $\mathrm{HH}^{n,s}(\mathrm{N}_m(R), M) = H^n(C^{*,s}(\mathrm{N}_m(R), M))$.

The filtration $F^pM$ is compatible with the $\mathbb{Z}$-grading. Put $(F^pM)^s = F^pM \cap M^s$. By the $\mathbb{Z}$-grading, we have the degree $s$ component of the spectral sequence $\{E_r^{p,q}(\mathrm{N}_m(R), M), d_r\}_{r\geq 1}$:

$$E_1^{p,q,s}(\mathrm{N}_m(R), M) \Longrightarrow \mathrm{HH}^{p+q,s}(\mathrm{N}_m(R), M),$$

where $E_1^{p,q,s}(\mathrm{N}_m(R), M) \cong \mathrm{HH}^{p+q,s}(\mathrm{N}_m(R), F^pM/F^{p+1}M)$, $d_r^s : E_r^{p,q,s}(\mathrm{N}_m(R), M) \to E_r^{p+r,q-r+1,s}(\mathrm{N}_m(R), M)$, $E_r^{p,q}(\mathrm{N}_m(R), M) = \oplus_{s\in\mathbb{Z}} E_r^{p,q,s}(\mathrm{N}_m(R), M)$, and $d_r = \oplus_{s\in\mathbb{Z}} d_r^s$.

**Proposition 8.** *When $M = \mathrm{N}_m(R)$ or $\mathrm{M}_m(R)/\mathrm{N}_m(R)$,*

$$E_1^{p,q,s}(\mathrm{N}_m(R), M) = 0$$

*if $s \neq q$.*

We omit the proof of Proposition 8. For details, see [2].

By Proposition 8, we have the following corollary.

**Corollary 9.** *When* $M = \mathrm{N}_m(R)$ *or* $\mathrm{M}_m(R)/\mathrm{N}_m(R)$, *the spectral sequence*

$$E_1^{p,q}(\mathrm{N}_m(R), M) \Longrightarrow \mathrm{HH}^*(\mathrm{N}_m(R), M)$$

*collapses from* $E_2$*-page and there is no extension problem.*

## 5. Step 3

By Step 2, we only need to calculate $E_2^{p,q}(\mathrm{N}_m(R), M)$ for determining the $R$-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), M)$ when $M = \mathrm{N}_m(R)$ or $\mathrm{M}_m(R)/\mathrm{N}_m(R)$.

Since $F^pM/F^{p+1}M$ is isomorphic to the direct sum of $R$ as $\mathrm{N}_m(R)$-bimodules over $R$,

$$\begin{aligned} E_1^{p,q}(\mathrm{N}_m(R), M) &\cong \mathrm{HH}^{p+q}(\mathrm{N}_m(R), F^pM/F^{p+1}M) \\ &\cong \mathrm{HH}^{p+q}(\mathrm{N}_m(R), R) \otimes_R (F^pM/F^{p+1}M) \\ &\cong \mathrm{N}_m(R)^!_{p+q} \otimes_R (F^pM/F^{p+1}M) \end{aligned}$$

by Step 1.

The differential $d_1 : E_1^{p,q}(\mathrm{N}_m(R), M) \to E_1^{p+1,q}(\mathrm{N}_m(R), M)$ can be identified with the connecting homomorphism

$$\mathrm{HH}^{p+q}(\mathrm{N}_m(R), F^pM/F^{p+1}M) \to \mathrm{HH}^{p+q+1}(\mathrm{N}_m(R), F^{p+1}M/F^{p+2}M)$$

induced by

$$0 \to F^{p+1}M/F^{p+2}M \to F^pM/F^{p+2}M \to F^pM/F^{p+1}M \to 0.$$

The connecting homomorphism can be described explicitly.

Recall $\varphi(d) = \mathrm{rank}_R \mathrm{N}_m(R)^!_d$. The following propositions can be proved by long discussions. For details, see [2].

**Proposition 10.** *Let* $m \geq 3$. *For* $p \neq 0$, $E_2^{p,q}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = 0$. *For* $p = 0$, $E_2^{0,q}(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ *is a free* $R$*-module of rank*

$$(m-1)\varphi(q) + \sum_{k=1}^{m-1} (-1)^{m+k} k\varphi(q-m+k).$$

**Proposition 11.** *For $p \neq 0, 1, m-1$, $E_2^{p,q}(\mathrm{N}_m(R), \mathrm{N}_m(R)) = 0$. For $p = 0, 1, m-1$, $E_2^{p,q}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is a free $R$-module. The rank of $E_2^{p,q}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ is given by*

$$\begin{aligned}
\mathrm{rank}_R E_2^{0,q} &= \begin{cases} 1 & (q = 0) \\ 0 & (q \neq 0), \end{cases} \\
\mathrm{rank}_R E_2^{1,q} &= \begin{cases} m-1 & (q = 0) \\ (m-2)\varphi(q) & (q \neq 0), \end{cases} \\
\mathrm{rank}_R E_2^{m-1,q} &= (-1)^m \varphi(q) + \sum_{k=0}^{m-1} (-1)^k (k+1) \varphi(q+m-k-1).
\end{aligned}$$

Since there is no extension problem, we can determine the $R$-module structure of $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ and $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ (Theorems 1 and 2). Hence, we have proved our main theorem except for the product structure on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$.

## 6. Step 4

In this section, we determine the product structure on $E_\infty^{p,q}$ for $M = \mathrm{N}_m(R)$.

Let $\mathcal{A}$ be an abelian symmetric monoidal category in which the tensor product $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is right exact separately in each variable.

**Definition 12.** Let $(A^*, d)$ be a differential graded algebra in $\mathcal{A}$. Suppose that we have a filtration

$$A^* = F^0 A^* \supset F^1 A^* \supset \cdots \supset F^n A^* \supset \cdots \supset F^t A^* = 0.$$

A triple $(A^*, d, \{F^r A^*\}_{r \geq 0})$ is said to be a filtered differential graded algebra if it satisfies the following two conditions:

(1) For any $n \geq 0$, $d(F^n A^*) \subset F^n A^*$.
(2) For any $r, s \geq 0$, $F^r A^* \cdot F^s A^* \subset F^{r+s} A^*$.

Let $(A^*, d, \{F^r A^*\}_{r \geq 0})$ be a filtered differential graded algebra. By [3, Theorem 2.14], there is a spectral sequence

$$E_1^{p,q} = H^{p+q}(F^p A / F^{p+1} A) \Longrightarrow H^{p+q}(A)$$

of algebras in $\mathcal{A}$, which converges to $H^{p+q}(A)$ as an algebra.

Recall the decomposition

$$C^p(\mathrm{N}_m(R), \mathrm{N}_m(R)) = \oplus_{s \in \mathbb{Z}} C^{p,s}(\mathrm{N}_m(R), \mathrm{N}_m(R))$$

as a $\mathbb{Z}$-graded $R$-module, which is compatible with the filtration $F^r C^*(\mathrm{N}_m(R), \mathrm{N}_m(R)) = C^*(\mathrm{N}_m(R), F^r \mathrm{N}_m(R))$. Then the triple $(C^*(\mathrm{N}_m(R), \mathrm{N}_m(R)), d, \{F^r C^*(\mathrm{N}_m(R), \mathrm{N}_m(R))\}_{r \geq 0})$ is a filtered differential graded algebra in the category of $\mathbb{Z}$-graded $R$-modules.
Thus, we obtain a multiplicative spectral sequence

$$E_1^{p,q}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{N}_m(R), \mathrm{N}_m(R))$$

in the abelian category of $\mathbb{Z}$-graded $R$-modules.

In the case $m = 3$, we can determine the product structure on $E_\infty^{p,q}$ and $\mathrm{HH}^*(\mathrm{N}_3(R), \mathrm{N}_3(R))$ directly. Here we assume that $m \geq 4$. The following lemma is essential for determining the product structure on $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$.

**Lemma 13.** *For $m \geq 4$, let $a \in \mathrm{HH}^{1+q,q}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ and $b \in \mathrm{HH}^{1+q',q'}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ represented by $x \in E_\infty^{1,q,q}$ and $y \in E_\infty^{1,q',q'}$, respectively. Then we obtain $ab = 0$ in $\mathrm{HH}^{2+q+q',q+q'}(\mathrm{N}_m(R), \mathrm{N}_m(R))$.*

*Proof.* Since $E_2^{2,q+q',q+q'} = 0$ for $m \geq 4$ by Proposition 11, $E_\infty^{2,q+q',q+q'} = 0$. Hence $xy = 0$, which implies that $ab$ is represented by an element in $E_\infty^{m-1,q+q'-m+3,q+q'}$. By Proposition 8, if $m \geq 4$, then $E_\infty^{m-1,q+q'-m+3,q+q'} = E_1^{m-1,q+q'-m+3,q+q'} = 0$. Therefore $ab = 0$. $\square$

By Lemma 13, we can prove Theorem 3. For details, see [2].

## References


[1] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2), **78** (1963), 267–288.

[2] T. Itagaki, K. Nakamoto, and T.Torii, *Hochschild cohomology of* $\mathrm{N}_m$, in preparation.

[3] J. McCleary, *A user's guide to spectral sequences*, Second edition, Cambridge Studies in Advanced Mathematics, 58, Cambridge University Press, Cambridge, 2001.

[4] K. Nakamoto and T. Torii, Applications of Hochschild cohomology to the moduli of subalgebras of the full matrix ring, preprint, 2020, arXiv:2006.07870.

[5] E. Sköldberg, The Hochschild homology of truncated and quadratic monomial algebras, J. London Math. Soc. (2) **59** (1999), no.1, 76–86.

[6] S. J. Witherspoon, *Hochschild cohomology for algebras*, Graduate Studies in Mathematics, 204, American Mathematical Society, Providence, RI, 2019.



Department of Economics
Faculty of Economics
Takasaki City University of Economics
1300 Kaminamie, Takasaki, Gunma, 370-0801, JAPAN
*E-mail address*: titagaki@tcue.ac.jp

Center for Medical Education and Sciences
Faculty of Medicine
University of Yamanashi
Chuo, Yamanashi 409-3898 JAPAN
*E-mail address*: nakamoto@yamanashi.ac.jp

Department of Mathematics
Okayama University
Okayama 700-8530 JAPAN
*E-mail address*: torii@math.okayama-u.ac.jp