CLASSIFYING SUBCATEGORIES OF NOETHERIAN ALGEBRAS

OSAMU IYAMA AND YUTA KIMURA

ABSTRACT. For a Noetherian R-algebra Λ, we classify torsion classes, torsionfree classes and Serre subcategories of $\mathsf{mod}\,\Lambda$. For a prime ideal $\mathfrak{p}$ of R, let $k_{\mathfrak{p}}\Lambda = (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \otimes_R \Lambda$. Torsionfree classes are classified by using torsionfree classes of $k_{\mathfrak{p}}\Lambda$. Serre subcategories are classified by using simple $k_{\mathfrak{p}}\Lambda$-modules. To classify torsion classes, we construct an embedding from $\mathsf{tors}\,\Lambda$ to $\prod_{\mathfrak{p}} \mathsf{tors}\, k_{\mathfrak{p}}\Lambda$, where $\mathsf{tors}\,\Lambda$ is the set of torsion classes of $\mathsf{mod}\,\Lambda$ and $\mathfrak{p}$ runs all prime ideals of R. We introduce the notion of compatible elements in $\prod_{\mathfrak{p}} \mathsf{tors}\, k_{\mathfrak{p}}\Lambda$ and show that each element in the image of the embedding is compatible. We give a sufficient condition such that any compatible element belongs to the image of the embedding. This proceeding is based on the paper [6].

1. Preliminary

For finite dimensional algebras, a connection between torsion classes and classical tilting modules was well understood in the last century, see [2] for instance. Recently, there are many studies of torsion classes, for instance [1, 5]. For a commutative Noetherian ring R, classification problems of subcategories of $\mathsf{mod}\,R$ has been studied by many mathematicians. The classification of Serre subcategories by Gabriel [3] is one of the most important results. There exist many results of classification problems of subcategories based on the Gabriel's result, for instance [7, 13].

Throughout this proceeding let R be a commutative Noetherian ring and Λ an R-algebra which is finitely generated as an R-module. We call such an algebra Λ a *Noetherian algebra* and write (R, Λ). In the paper [6], as a natural generalization of finite dimensional algebras and commutative Noetherian rings, we consider Noetherian R-algebras. The aim is to classify torsion classes, torsionfree classes and Serre subcategories of the category $\mathsf{mod}\,\Lambda$ of finitely generated (left) Λ-modules.

We recall the definition of such subcategories of the module category.

Definition 1. Let (R, Λ) be a Noetherian algebra and $\mathcal{C}$ a subcategory of $\mathsf{mod}\,\Lambda$. We say that $\mathcal{C}$ is a *torsion class* (respectively, *torsionfree class*, *Serre subcategory*) of $\mathsf{mod}\,\Lambda$ if $\mathcal{C}$ is closed under factor modules (respectively, submodules, both of factor modules and submodules) and extensions. We denote by $\mathsf{tors}\,\Lambda$ (respectively, $\mathsf{torf}\,\Lambda$, $\mathsf{serre}\,\Lambda$) the set of all torsion classes (respectively, torsionfree classes, Serre subcategories) of $\mathsf{mod}\,\Lambda$.

Note that each torsion class in $\mathsf{mod}\,\Lambda$ gives rise to a torsion pair, since each module in $\mathsf{mod}\,\Lambda$ is noetherian. On the other hand, a torsionfree class does not necessarily give rise to a torsion pair in $\mathsf{mod}\,\Lambda$.

We denote by $\mathsf{Spec}R$ the set of all prime ideals of R. Let $\mathfrak{p} \in \mathsf{Spec}R$ and $\Lambda_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R \Lambda$. Then $(R_{\mathfrak{p}}, \Lambda_{\mathfrak{p}})$ is a Noetherian algebra. For a Λ-module M we denote by $M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M$

The detailed version of this paper will be submitted for publication elsewhere.

a $\Lambda_{\mathfrak{p}}$-module. For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, we denote by $\mathcal{C}_{\mathfrak{p}}$ a subcategory of $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$ defined as follows:

$$\mathcal{C}_{\mathfrak{p}} := \{M_{\mathfrak{p}} \in \mathsf{mod}\,\Lambda_{\mathfrak{p}} \mid M \in \mathcal{C}\}.$$

It is easy to see that if $\mathcal{C}$ is closed under extensions (respectively, factor modules, submodules) in $\mathsf{mod}\,\Lambda$, then so is $\mathcal{C}_{\mathfrak{p}}$ in $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$. Therefore taking localization at $\mathfrak{p}$ preserves the property of being torsion classes (respectively, torsionfree classes, Serre subcategories). Thus:

Lemma 2. *If $\mathcal{C}$ is a torsion class (respectively, torsionfree class, Serre subcategory) of $\mathsf{mod}\,\Lambda$, then $\mathcal{C}_{\mathfrak{p}}$ is a torsion class (respectively, torsionfree class, Serre subcategory) of $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$.*

For $\mathfrak{p} \in \mathsf{Spec}R$ let $k_{\mathfrak{p}} = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $k_{\mathfrak{p}}\Lambda = k_{\mathfrak{p}} \otimes_R \Lambda$. Then $k_{\mathfrak{p}}\Lambda$ is a finite dimensional $k_{\mathfrak{p}}$-algebra. We regard $\mathsf{mod}\,k_{\mathfrak{p}}\Lambda$ as a full subcategory of $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$ by a canonical surjection from $\Lambda_{\mathfrak{p}}$ to $k_{\mathfrak{p}}\Lambda$. Then $\mathsf{mod}\,k_{\mathfrak{p}}\Lambda$ is closed under factor modules and submodules in $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$. Thus an assignment $\mathcal{C} \mapsto \mathcal{C} \cap \mathsf{mod}\,k_{\mathfrak{p}}\Lambda$ induces maps

$$\text{(1.1)} \qquad \mathsf{tors}\,\Lambda_{\mathfrak{p}} \longrightarrow \mathsf{tors}\,k_{\mathfrak{p}}\Lambda, \qquad \mathsf{torf}\,\Lambda_{\mathfrak{p}} \longrightarrow \mathsf{torf}\,k_{\mathfrak{p}}\Lambda, \qquad \mathsf{serre}\,\Lambda_{\mathfrak{p}} \longrightarrow \mathsf{serre}\,k_{\mathfrak{p}}\Lambda.$$

Let $\mathbb{T}_R(\Lambda)$, $\mathbb{F}_R(\Lambda)$ and $\mathbb{S}_R(\Lambda)$ be the Cartesian products of $\mathsf{tors}\,k_{\mathfrak{p}}\Lambda$, $\mathsf{torf}\,k_{\mathfrak{p}}\Lambda$ and $\mathsf{serre}\,k_{\mathfrak{p}}\Lambda$ respectively, where $\mathfrak{p}$ runs all prime ideals of R:

$$\mathbb{T}_R(\Lambda) := \prod_{\mathfrak{p}\in\mathsf{Spec}R} \mathsf{tors}\,k_{\mathfrak{p}}\Lambda, \qquad \mathbb{F}_R(\Lambda) := \prod_{\mathfrak{p}\in\mathsf{Spec}R} \mathsf{torf}\,k_{\mathfrak{p}}\Lambda, \qquad \mathbb{S}_R(\Lambda) := \prod_{\mathfrak{p}\in\mathsf{Spec}R} \mathsf{serre}\,k_{\mathfrak{p}}\Lambda.$$

By Lemma 2 and (1.1), we have the following maps

$$\Phi : \mathsf{tors}\,\Lambda \longrightarrow \mathbb{T}_R(\Lambda), \qquad \mathcal{T} \mapsto \{\mathcal{T}_{\mathfrak{p}} \cap \mathsf{mod}\,k_{\mathfrak{p}}\Lambda\}_{\mathfrak{p}},$$
$$\Phi' : \mathsf{torf}\,\Lambda \longrightarrow \mathbb{F}_R(\Lambda), \qquad \mathcal{F} \mapsto \{\mathcal{F}_{\mathfrak{p}} \cap \mathsf{mod}\,k_{\mathfrak{p}}\Lambda\}_{\mathfrak{p}}.$$

By restricting Φ to $\mathsf{serre}\,\Lambda$, we have a map from $\mathsf{serre}\,\Lambda$ to $\mathbb{S}_R(\Lambda)$. These maps enable us to study torsion classes, torsionfree classes and Serre subcategories of $\mathsf{mod}\,\Lambda$ by comparing with those of $\mathsf{mod}\,k_{\mathfrak{p}}\Lambda$.

2. Classification of torsionfree classes and Serre subcategories

We regard $\mathsf{tors}\,\Lambda$, $\mathsf{torf}\,\Lambda$, $\mathsf{serre}\,\Lambda$, $\mathbb{T}_R(\Lambda)$, $\mathbb{F}_R(\Lambda)$ and $\mathbb{S}_R(\Lambda)$ as posets by inclusion.

For a subcategory $\mathcal{C}$ of $\mathsf{mod}\,\Lambda$, let $\mathcal{C}^{\perp}$ be a subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules M such that $\mathrm{Hom}_{\Lambda}(C, M) = 0$ for any $C \in \mathcal{C}$. Dually we define ${}^{\perp}\mathcal{C}$. Since $k_{\mathfrak{p}}\Lambda$ is a finite dimensional algebra, $(-)^{\perp}$ induces an order reversing bijection from $\mathsf{tors}\,k_{\mathfrak{p}}\Lambda$ to $\mathsf{torf}\,k_{\mathfrak{p}}\Lambda$ with an inverse map ${}^{\perp}(-)$. Then $(-)^{\perp}$ induces an order reversing bijection from $\mathbb{T}_R(\Lambda)$ to $\mathbb{F}_R(\Lambda)$. We have the following commutative diagram.

$$\begin{array}{ccc} \mathsf{tors}\,\Lambda & \xrightarrow{\Phi} & \mathbb{T}_R(\Lambda) \\ {\scriptstyle (-)^{\perp}}\downarrow & & {\scriptstyle (-)^{\perp}}\downarrow\wr \\ \mathsf{torf}\,\Lambda & \xrightarrow{\Phi'} & \mathbb{F}_R(\Lambda) \end{array}$$

Our main theorem is the following one.

Theorem 3. *For a Noetherian algebra (R, Λ), the following statements hold.*

(a) *The map* Φ' *is an isomorphism of posets.*
(b) *The map* Φ *is an embedding of posets.*

Therefore classification problem of torsionfree classes of $\mathsf{mod}\,\Lambda$ can be reduced to the problem for finite dimensional algebras. As we explain in Corollary 5 bellow if $\Lambda = R$ then Theorem 3 recovers famous classification results of torsion classes, torsionfree classes and Serre subcategories by Stanley-Wang, Takahashi and Gabriel, respectively [3, 12, 13].

We give an inverse map of Φ'. We denote by $\operatorname{Ass} M$ the set of associated prime ideals of an R-module M. For a torsionfree class $\mathcal{Y}$ of $\mathsf{mod}\, k_{\mathfrak{p}}\Lambda$ let

$$\check{\mathcal{Y}} := \{X \in \mathsf{mod}\,\Lambda \mid \operatorname{Ass} X \subseteq \{\mathfrak{p}\},\ X_{\mathfrak{p}} \in \mathsf{F}_{\Lambda_{\mathfrak{p}}}(\mathcal{Y})\}.$$

where $\mathsf{F}_{\Lambda_{\mathfrak{p}}}(\mathcal{Y})$ is the smallest torsionfree class of $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$ containing $\mathcal{Y}$. Then the inverse map Ψ' of Φ' is given as follows

$$\Psi' : \mathbb{F}_R(\Lambda) \longrightarrow \mathsf{torf}\,\Lambda, \qquad \{\mathcal{Y}^{\mathfrak{p}}\}_{\mathfrak{p}} \mapsto \mathsf{Filt}\left(\widecheck{\mathcal{Y}^{\mathfrak{p}}} \middle| \mathfrak{p} \in \mathsf{Spec} R\right).$$

We apply Theorem 3 to obtain a classification of Serre subcategories of $\mathsf{mod}\,\Lambda$. We denote by $\mathsf{sim}\, k_{\mathfrak{p}}\Lambda$ the set of isomorphism classes of simple $k_{\mathfrak{p}}\Lambda$-modules and let

$$\mathsf{Sim} := \bigsqcup_{\mathfrak{p} \in \mathsf{Spec} R} \mathsf{sim}\, k_{\mathfrak{p}}\Lambda.$$

We regard Sim as posets as follows: for $S \in \mathsf{sim}\, k_{\mathfrak{p}}\Lambda$ and $T \in \mathsf{sim}\, k_{\mathfrak{q}}\Lambda$, we write $S \leqslant T$ if and only if $\mathfrak{p} \supseteq \mathfrak{q}$ and S is a subfactor of T as a $\Lambda_{\mathfrak{q}}$-module. We say that a subset $\mathcal{W}$ of Sim is a *down-set* if $T \in \mathcal{W}$ and $S \leqslant T$ implies $S \in \mathcal{W}$ for $S \in \mathsf{Sim}$.

It is well-known that an assignment $\mathcal{C} \mapsto \mathcal{C} \cap \mathsf{sim}\, k_{\mathfrak{p}}\Lambda$ induces an isomorphism of posets from $\mathsf{serre}\, k_{\mathfrak{p}}\Lambda$ to the power set $\mathsf{P}(\mathsf{sim}\, k_{\mathfrak{p}}\Lambda)$. This induces an isomorphism of posets $\mathbb{S}_R(\Lambda) \simeq \mathsf{P}(\mathsf{Sim})$. Since the map Φ restricts to Serre subcategories, we have the following morphisms of posets

$$\mathsf{serre}\,\Lambda \longrightarrow \operatorname{Im}\Phi \subset \mathbb{S}_R(\Lambda) \simeq \mathsf{P}(\mathsf{Sim}).$$

We regard Φ as a map from $\mathsf{serre}\,\Lambda$ to $\mathsf{P}(\mathsf{Sim})$. Then the following theorem characterizes $\operatorname{Im}\Phi$.

Theorem 4. *For a Noetherian algebra* (R, Λ)*, the map* Φ *induces an isomorphism of posets*

$$\mathsf{serre}(\Lambda) \simeq \{\mathcal{W} \subseteq \mathsf{Sim} \mid \mathcal{W} \text{ is a down-set}\}$$

Note that this result simplifies Kanda's classification [7, 8] of $\mathsf{serre}\,\Lambda$ in terms of atom spectrum.

We give another application of Theorem 3. We say that a subset $\mathcal{W}$ of $\mathsf{Spec} R$ is *specialization closed* if $\mathfrak{p} \in \mathcal{W}$ and $\mathfrak{p} \subset \mathfrak{q}$ implies $\mathfrak{q} \in \mathcal{W}$ for $\mathfrak{q} \in \mathsf{Spec} R$. If we take $\Lambda = R$, then Theorem 3 recovers famous classification results of torsion classes, torsionfree classes and Serre subcategories by Stanley-Wang, Takahashi and Gabriel, respectively [3, 12, 13]. For $\mathcal{X} = \{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathbb{T}_R(\Lambda)$ (or $\mathbb{F}_R(\Lambda)$), let $\mathcal{S}(\mathcal{X}) = \{\mathfrak{p} \in \mathsf{Spec} R \mid \mathcal{X}^{\mathfrak{p}} \neq 0\}$. Then we have the following corollary.

Corollary 5. *Let* (R, Λ) *be a Noetherian algebra. Assume that* $\Lambda_{\mathfrak{p}}$ *is Morita equivalent to a local ring for each* $\mathfrak{p} \in \mathsf{Spec} R$*. Then the following statements hold.*

(a) *We have* $\mathsf{serre}\,\Lambda = \mathsf{tors}\,\Lambda$.

(b) *The composite* $\mathcal{S}\circ\Phi$ *is an isomorphism of posets and* $(\mathcal{S}\circ\Phi)(\mathcal{C})=\bigcup_{M\in\mathcal{C}}\operatorname{Supp}M$ *holds.*

$$\mathsf{serre}\,\Lambda\xrightarrow{\Phi}\mathrm{Im}\Phi\xrightarrow{\mathcal{S}}\{\textit{specialization closed subsets of } \mathsf{Spec}R\}$$

(c) *The composite* $\mathcal{S}\circ\Phi'$ *is an isomorphism of posets and* $(\mathcal{S}\circ\Phi')(\mathcal{C})=\bigcup_{M\in\mathcal{C}}\operatorname{Ass}M$ *holds.*

$$\mathsf{torf}\,\Lambda\xrightarrow{\Phi'}\mathbb{F}_R(\Lambda)\xrightarrow{\mathcal{S}}\mathsf{P}(\mathsf{Spec}R)$$

3. Classification of torsion classes

The map Φ is an embedding of posets from $\mathsf{tors}\,\Lambda$ to $\mathbb{T}_R(\Lambda)$ by Theorem 3. Thus we study the subset $\mathrm{Im}\Phi$ of $\mathbb{T}_R(\Lambda)$. For $\mathcal{T}\in\mathsf{tors}\,k_{\mathfrak{p}}\Lambda$, the following subcategory $\overline{\mathcal{T}}$ is a torsion class of $\mathsf{mod}\,\Lambda_{\mathfrak{p}}$:

$$\overline{\mathcal{T}}=\{X\in\mathsf{mod}\,\Lambda_{\mathfrak{p}}\mid X/\mathfrak{p}X\in\mathcal{T}\}\in\mathsf{tors}\,\Lambda_{\mathfrak{p}}.$$

For $\mathfrak{p}\supseteq\mathfrak{q}$ of $\mathsf{Spec}R$, we define a map $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}$ by the composite of the following three maps

$$\mathrm{r}_{\mathfrak{p},\mathfrak{q}}:\mathsf{tors}\,k_{\mathfrak{p}}\Lambda\xrightarrow{\overline{(-)}}\mathsf{tors}\,\Lambda_{\mathfrak{p}}\xrightarrow{(-)_{\mathfrak{q}}}\mathsf{tors}\,\Lambda_{\mathfrak{q}}\xrightarrow{(-)\cap\mathsf{mod}\,k_{\mathfrak{q}}\Lambda}\mathsf{tors}\,k_{\mathfrak{q}}\Lambda.$$

Definition 6. We say that $\{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}}\in\mathbb{T}_R(\Lambda)$ is *compatible* if $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}(\mathcal{X}^{\mathfrak{p}})\supseteq\mathcal{X}^{\mathfrak{q}}$ holds for any prime ideals $\mathfrak{p}\supseteq\mathfrak{q}$.

We can show the following proposition.

Proposition 7. *Any element of* $\mathrm{Im}\Phi$ *is compatible.*

We say that a Noetherian algebra (R,Λ) is *compatible* if any compatible element of $\mathbb{T}_R(\Lambda)$ belongs to $\mathrm{Im}\Phi$. In this case, we have

$$\mathsf{tors}\,\Lambda\simeq\left\{\{\mathcal{X}^{\mathfrak{p}}\}_{\mathfrak{p}}\in\prod_{\mathfrak{p}\in\mathsf{Spec}R}\mathsf{tors}\,k_{\mathfrak{p}}\Lambda\;\middle|\;\mathrm{r}_{\mathfrak{p},\mathfrak{q}}(\mathcal{X}^{\mathfrak{p}})\supseteq\mathcal{X}^{\mathfrak{q}},{}^{\forall}\mathfrak{p}\supseteq\mathfrak{q}\in\mathsf{Spec}R\right\}$$

which gives a complete classification of $\mathsf{tors}\,\Lambda$.

There are many Noetherian algebra which are compatible. For example we have the following theorem.

Theorem 8. *Let* (R,Λ) *be a Noetherian algebra. If* R *is semi-local with Krull dimension one, then* (R,Λ) *is compatible.*

Note that the classification of torsion classes was also studied in [9] in the case when R is a complete local domain with Krull dimension one.

We give another example. Let k be a field and A a finite dimensional k-algebra. A simple A-module S is said to be *k-simple* if $\mathrm{End}_A(S)\simeq k$ holds. For instance if k is algebraically closed, or A is a factor algebra of a finite quiver modulo an admissible ideal, then all simple A-modules are k-simple.

Theorem 9. *Let* A *be a finite dimensional* k*-algebra and* R *a commutative Noetherian ring containing a field* k*. Assume that all simple* A*-modules are* k*-simple and* $\mathsf{tors}\,A$ *is a finite set. Then the following statements hold.*

(a) *There exists an isomorphism of posets* $\mathrm{t}_{\mathfrak{p}}:\mathsf{tors}\,A\to\mathsf{tors}(k_{\mathfrak{p}}\otimes_k A)$ *such that* $\mathrm{r}_{\mathfrak{p},\mathfrak{q}}\circ\mathrm{t}_{\mathfrak{p}}=\mathrm{t}_{\mathfrak{q}}$ *holds for any prime ideals* $\mathfrak{p}\supseteq\mathfrak{q}$.

(b) *The Noetherian algebra* $(R, R \otimes_k A)$ *is compatible.*
(c) *We have an isomorphism of posets*

$$\mathsf{tors}(R \otimes_k A) \simeq \mathrm{Hom}_{\mathrm{poset}}(\mathsf{Spec} R, \mathsf{tors}\, A).$$

We give one basic example.

Example 10. Let Q be a Dynkin quiver, and R a commutative Noetherian ring which contains a field k. It is known that $\mathsf{tors}(kQ)$ is isomorphic to the Cambrian lattice $\mathfrak{C}_Q$ of Q by [5] and [11]. Since $RQ \simeq R \otimes_k kQ$, we have the following isomorphism of posets by Theorem 9.

$$\mathsf{tors}\, RQ \simeq \mathrm{Hom}_{\mathrm{poset}}(\mathsf{Spec} R, \mathfrak{C}_Q).$$

We end this proceeding posing the following question.

Question 11. Characterize Noetherian algebras which are compatible.

So far we do not know any Noetherian algebra which is *not* compatible.

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Osamu Iyama
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914, JAPAN
Email address: iyama@ms.u-tokyo.ac.jp

Yuta Kimura
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914, JAPAN
Email address: ykimura@ms.u-tokyo.ac.jp