STRUCTURE THEOREM FOR FLAT COTORSION MODULES OVER NOETHER ALGEBRAS

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Abstract. For a module-finite algebra over a commutative noetherian ring, we give a complete description of flat cotorsion modules in terms of prime ideals of the algebra, as a generalization of Enochs' result for a commutative noetherian ring. As a consequence, we show that pointwise Matlis duality gives a bijective correspondence between the isoclasses of indecomposable flat cotorsion right modules and the isoclasses of indecomposable injective left modules. This correspondence is an explicit realization of Herzog's homeomorphism between Ziegler spectra, which was given in terms of elementary duality.

Key Words: Flat cotorsion module; Noether algebra; ideal-adic completion.

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1. Introduction

A right module M over a ring A is called *cotorsion* if $\mathrm{Ext}^1_A(F, M) = 0$ for all flat right A-modules F. A *flat cotorsion* module is a module that is flat and cotorsion. The *flat cover conjecture*, which was affirmatively solved by Bican, El Bashir, and Enochs [BEBE01], implies that the class of flat modules and the class of cotorsion modules form a complete cotorsion pair. Flat cotorsion modules are those modules that belong to the core of this cotorsion pair.

Enochs [Eno84] gave a structure theorem for flat cotorsion modules over a commutative noetherian ring R: An R-module M is flat cotorsion if and only if M is isomorphic to

$$\prod_{\mathfrak{p}\in\operatorname{Spec} R} \mathrm{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p})^{(B_\mathfrak{p})})$$

for some family of sets $\{B_\mathfrak{p}\}_{\mathfrak{p}\in\operatorname{Spec} R}$, where $E_R(R/\mathfrak{p})$ is the injective envelope of $R/\mathfrak{p}$ and $E_R(R/\mathfrak{p})^{(B_\mathfrak{p})}$ is the direct sum of its $B_\mathfrak{p}$-indexed copies. The cardinality of each $B_\mathfrak{p}$ is uniquely determined by the isomorphism class of M.

In this paper, we state our result that generalizes Enochs' structure theorem to Noether R-algebras. Details of our result can be found in [KN21].

This is a summary of [KN21]. The detailed version of this paper will be submitted for publication elsewhere.

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2. Structure theorem

For a commutative noetherian ring R, a *Noether R-algebra* is a ring A together with a ring homomorphism $\varphi\colon R \to A$ such that the image of φ is contained in the center of A and A is finitely generated as an R-module. $\operatorname{Spec} A$ denotes the set of prime (two-sided) ideals of A.

For each $P \in \operatorname{Spec} A$, denote by $I_A(P)$ the corresponding indecomposable injective right A-module, whose only associated prime is P. Note that $P \cap R := \varphi^{-1}(P)$ is a prime ideal of R.

The following is our main result:

Theorem 1. *Let A be a Noether R-algebra. A right A-module M is flat cotorsion if and only if M is isomorphic to*

$$\prod_{P \in \operatorname{Spec} A} \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/(P \cap R))^{(B_P)})$$

for some family of sets $\{B_P\}_{P \in \operatorname{Spec} A}$. The cardinality of each B_P is uniquely determined by the isomorphism class of M.

Corollary 2. *Let A be a Noether R-algebra. There is a bijection*

$$\operatorname{Spec} A \xrightarrow{\sim} \{\text{isoclasses of indecomposable flat cotorsion right } A\text{-modules}\}$$

given by $P \mapsto T_A(P) := \operatorname{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/P \cap R))$.

The following result is useful to classify flat cotorsion modules over concrete algebras:

Proposition 3. *For every $\mathfrak{p} \in \operatorname{Spec} R$, there is an isomorphism*

$$\widehat{A_{\mathfrak{p}}} \cong \bigoplus_{\substack{P \in \operatorname{Spec} A \\ P \cap R = \mathfrak{p}}} T_A(P)^{n_P}$$

of right A-modules, where each n_P is a (finite) positive number.

Example 4. Let R be a commutative noetherian ring and let

$$A := \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

Then A is a Noether R-algebra. For each $\mathfrak{p} \in \operatorname{Spec} R$, we have an indecomposable decomposition

$$\widehat{A_{\mathfrak{p}}} = \begin{pmatrix} \widehat{R_{\mathfrak{p}}} & 0 \\ \widehat{R_{\mathfrak{p}}} & \widehat{R_{\mathfrak{p}}} \end{pmatrix} \cong \begin{pmatrix} \widehat{R_{\mathfrak{p}}} & 0 \end{pmatrix} \oplus \begin{pmatrix} \widehat{R_{\mathfrak{p}}} & \widehat{R_{\mathfrak{p}}} \end{pmatrix}$$

as a right A-module. Therefore

$$\begin{pmatrix} \widehat{R_{\mathfrak{p}}} & 0 \end{pmatrix}, \begin{pmatrix} \widehat{R_{\mathfrak{p}}} & \widehat{R_{\mathfrak{p}}} \end{pmatrix} \quad (\mathfrak{p} \in \operatorname{Spec} R)$$

are all the indecomposable flat cotorsion right A-modules up to isomorphism.

The correspondence $T_A(P) \leftrightarrow I_{A^{\mathrm{op}}}(P)$ gives an explicit realization of Herzog's homeomorphism between Ziegler spectra, which was given in terms of elementary duality. For more details, see [KN21, section 8].

References

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