ON ALMOST N-PROJECTIVE MODULES AND GENERALIZED N-PROJECTIVE MODULES

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Abstract. Almost N-projective modules and generalized N-projective modules play an important role in the study of lifting modules. In this paper, we consider a relationship between almost N-projective modules and generalized N-projective modules, and give new characterizations of these projectivities. by homomorphisms between their projective covers, respectively. Moreover, using the result, we consider a condition for a module which is almost N_i-projective for any $i \in I$ to be almost $\oplus_{i\in I} N_i$-projective.

1. Introduction

In 1960, Bass [3] introduced the notion of semiperfect rings, and three years later, Mares [11] defined that of semiperfect modules which is a generalization of semiperfect rings. In 1983, Oshiro [14] further generalized this module to (quasi-)semiperfect modules without the assumption of projective modules, and proved that every quasi-semiperfect module is a direct sum of hollow modules. (Quasi-)semiperfect modules defined by Oshiro are often called (quasi-)discrete modules. In 1989, Harada and Tozaki [6] introduced the notion of almost M-projective and, for direct sums of hollow modules, they study relationships between almost M-projective modules and modules which are related lifting modules. Let M and N be modules. M is called *almost N-projective* if for any submodule X of N and any homomorphism $f : M \to N/X$, either there exists a homomorphism $g : M \to N$ such that $\pi g = f$ or there exist a nonzero direct summand N_1 of N and a homomorphism $h : N_1 \to M$ such that $fh = \pi|_{N_1}$, where $\pi : N \to N/X$ is the natural epimorphism.

$$\begin{array}{cccc} & g & M & \\ & \swarrow & \circlearrowleft \downarrow f & \\ N & \overset{\pi}{\to} & N/X & \to 0 \end{array} \qquad \text{or} \qquad \begin{array}{cccc} N_1 & \overset{h}{\to} & M & \\ \iota \downarrow & \circlearrowleft & \downarrow f & \\ N & \overset{\pi}{\to} & N/X & \to 0 \end{array}$$

where ι is the canonical injection. It is known that if M is almost N-projective then M' is almost N'-projective for any direct summand M' of M and any submodule N' of N.

After that, Baba [1] introduced the notion of almost M-injective as dual to almost M-projective, and Baba and Harada [2] give necessary and sufficient conditions for a direct sum of hollow (resp. uniform) modules with a local endomorphism ring to be a lifting module (resp. an extending module). In 2002, Oshiro and his students [4] classified extending modules in terms of whether they satisfy the finite internal exchange property or not, and they introduced the notion of generalized N-injective and gave necessary and sufficient conditions for a direct sum of extending modules with the finite internal exchange property to be extending with the finite internal exchange property. Moreover, Mohamed

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and Müller [13] defined generalized N-projective as a dual concept of generalized N-injective to study of a direct sum of lifting modules. A module M is called *generalized N-projective* if for any submodule X of N and any homomorphism $f : M \to N/X$, there exist decompositions $M = M_1 \oplus M_2$, $N = N_1 \oplus N_2$, a homomorphism $g_1 : M_1 \to N_1$ and an epimorphism $g_2 : N_2 \to M_2$ such that $f|_{M_1} = \pi g_1$ and $\pi|_{N_2} = f g_2$, where $\pi : N \to N/X$ is the natural epimorphism.

$$\begin{array}{ccccccccc} M_1 & \oplus & M_2 & & = & & M & & \\ g_1 \downarrow & & g_2 \uparrow & & & & \downarrow f & & \\ N_1 & \oplus & N_2 & = & N & \stackrel{\pi}{\to} & N/X & \to 0 & . \end{array}$$

These projectivities play an important role in the study of a direct sum of lifting (hollow) modules. Clearly, if M is generalized N-projective, then it is almost N-projective. However even case that M and N are indecomposable modules over an artinian ring, the converse does not hold. Thus the following question is raised: "When are almost N-projective modules generalized N-projective?" In this paper, we give new characterizations of these projectivities and a condition for an almost N-projective module to be generalized N-projective.

Throughout this paper R is a ring with identity and modules are unitary right R-modules. $N \leq_{\oplus} M$ means that N is a direct summand of M. A submodule S of a module M is called *small* in M if $M \neq K + S$ for any proper submodule K of M and we write $S \ll M$ in this case. An epimorphism $f : A \to B$ is called *small epimorphism* if $\ker f \ll A$. A module M is said to be *lifting* if for any submodule X of M, there exists a direct summand M_1 of M such that $X/M_1 \ll M/M_1$. An indecomposable lifting module is called *hollow*. It is well known that R is a right (semi-)perfect ring if and only if any (finitely generated) projective R-module is lifting (cf. [12, Theorem 4.41]). Hence any (finitely generated) module over a right (semi-)perfect ring has the projective lifting cover. By [7, Theorem 8], any finite direct sum of projective lifting modules is also lifting. A lifting module M is said to be *quasi-discrete* if M satisfies the following condition: If A and B are direct summands of M such that $M = A + B$, then $A \cap B$ is a direct summand of M. Note that "quasi-projective lifting $\Rightarrow$ quasi-discrete"(cf. [12, Lemma 4.6]).

2. A relationship between almost N-projective modules and generalized N-projective modules

In this section, we consider a relationship between almost N-projective modules and generalized N-projective modules over any ring. Now we recall the graph. Let $M = A \oplus B$ and let $f : A \to B$ be a homomorphism. Then $\langle A \stackrel{f}{\to} B \rangle = \{a + f(a) \mid a \in A\}$ is a submodule of M which is called *the graph* with respect to $A \stackrel{f}{\to} B$. Note that $M = \langle A \stackrel{f}{\to} B \rangle \oplus B$. A family $\{X_i\}_{i \in I}$ of submodules of a module M is called a *local summand* of M if $\sum_{i \in I} X_i$ is direct and $\sum_{i \in F} X_i$ is a direct summand of M for any finite subset F of I. A module M is said to satisfy *LSS* if any local summand of M is a direct summand. It is well known that any module with LSS has an indecomposable decomposition ([14, Theorem 3.5] (cf. [12, Theorem 2.17])), and a module M satisfies LSS if and only if the union of any chain of direct summands of M is a summand ([12, Lemma 2.16]). Any lifting module over a right perfect ring satisfies LSS ([10, Theorem

3.3] or Proposition 1). The authors know no examples of a lifting module which does not satisfy LSS.

The following is a generalization of [10, Theorem 3.3].

Proposition 1. *Let M and N be lifting modules and let $f : M \to N$ be an epimorphism. If M satisfies LSS, then so is N.*

The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for every direct summand X of M, there exists a submodule N_i of M_i ($i \in I$) such that $M = X \oplus (\oplus_I N_i)$. A module M is said to have the (*finite*) *internal exchange property* (or, briefly, (F)IEP) if every (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable. This notion is introduced by Oshiro et al. [4]. It is known that any quasi-discrete module satisfies FIEP, and any lifting module with FIEP satisfies LSS ([14, Corollary 3.11], [9, Theorem 3.1]).

The following is a main result of this section.

Theorem 2. *Let M and N be lifting modules with LSS. Then M is almost N-projective if and only if M is generalized N-projective.*

By Theorem 2 and [8, Theorem 3.7], we see the following:

Corollary 3. *Let M_1 and M_2 be lifting modules with FIEP. Then Then $M_1 \oplus M_2$ is a lifting module with FIEP if and only if M_i is almost M_j-projective ($i \neq j$).*

3. A characterization of almost N-projective modules and its applications

In this section, we give a characterization of almost N-projective modules by homomorphisms between their projective covers.

Theorem 4. *Let M and N be modules with the projective lifting covers (P, ν_M) and (Q, ν_N), respectively (e.g. M and N are modules over a right perfect ring). Then the following two conditions are equivalent:*

(*a*) *M is almost N-projective.*

(*b*) *For any $\alpha \in \mathrm{Hom}_R(P, Q)$, either $\alpha(\ker \nu_M) \subseteq \ker \nu_N$, or there exist $P' \leq_\oplus P$ and $Q' \leq_\oplus Q$ such that $\alpha|_{P'} : P' \to Q'$ is an isomorphism, $(\alpha|_{P'})^{-1}(\ker \nu_N|_{Q'}) \subseteq \ker \nu_M|_{P'}$ and $0 \neq \nu_N(Q') \leq_\oplus N$.*

As an application of Theorem 4, we can obtain the following result which is a generalization of Harada's Theorem [5] in a sense.

Theorem 5. *Let M be a lifting module with the projective lifting cover, let N_i be a module with the projective lifting cover ($i \in I$). We consider the following conditions:*

(1) *M is almost N_i-projective and N_i is almost N_j-projective for any $i, j \in I$ ($i \neq j$).*

(2) *M is almost $\oplus_I N_i$-projective.*

Then (1) $\Rightarrow$ (2) holds. In particular, if each N_i is hollow, the decomposition $\oplus_{i \in I} N_i$ is exchangeable and M is not N_i-projective for any $i \in I$, then the converse holds.

Let M and N be modules with the projective lifting covers (P, ν_M) and (Q, ν_N), respectively. Then we note that M is N-projective if and only if, for any $\alpha \in \mathrm{Hom}_R(P, Q)$, $\alpha(\ker \nu_M) \subseteq \ker \nu_N$. Hence we can obtain " if M is N-projective then so is any closed submodule of M ". By this fact and Theorem 4, we can prove the following:

Proposition 6. *Let M, N_1 and N_2 be modules with the projective lifting covers. If M is N_1-projective and almost N_2-projective, then M is almost $N_1 \oplus N_2$-projective.*

The following is obtained from Proposition 1, Theorem 5 and Proposition 6.

Corollary 7. *(cf.* [5, Theorem]*) Let R be a right perfect ring, M a lifting module, N_i a hollow module ($i \in I$) and L_k a module ($k \in K$) such that (i) the decomposition $\oplus_{i\in I} N_i$ is exchangeable, (ii) M is almost N_i-projective but not N_i-projective for any $i \in I$, and (iii) M is L_k-projective for any $k \in K$. Then the following conditions are equivalent:*

(a) *M is almost $(\oplus_{i\in I} N_i) \oplus (\oplus_{k\in K} L_k)$-projective.*
(b) *N_i is almost N_j-projective for any distinct $i, j \in I$.*

At the end of this section, we shall give a generalization of [6, Proposition 4].

Corollary 8. *(cf.* [6, Proposition 4]*) Let M and N_i be modules with projective lifting covers. We assume M is almost N_i-projective for all $i \in I$. If P is uniserial, then M is almost $\oplus_{i\in I} N_i$-projective.*

4. A characterization of generalized N-projective modules and its applications

In this section, we first give a characterization of generalized projective modules by homomorphisms between their projective lifting covers. In addition, as its application, we consider a condition for a direct sum of lifting modules with the projective lifting covers to be lifting. The following is a main result of this section:

Theorem 9. *Let M and N be modules with projective lifting covers (P, ν_M) and (Q, ν_N), respectively. Then the following conditions are equivalent:*

(a) *M is generalized N-projective.*
(b) *For any $\alpha \in \mathrm{Hom}_R(P, Q)$, there exist decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $\alpha(P_1) \subseteq Q_1$, $\alpha(\ker \nu_M|_{P_1}) \subseteq \ker \nu_N|_{Q_1}$, $\alpha|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha|_{P_2})^{-1}(\ker \nu_N|_{Q_2}) \subseteq \ker \nu_M|_{P_2}$, $M = \nu_M(P_1) \oplus \nu_M(P_2)$ and $N = \nu_N(Q_1) \oplus \nu_N(Q_2)$.*
(c) *For any $\alpha \in \mathrm{Hom}_R(P, Q)$, there exist a decomposition $P = P_1 \oplus P_2$ and a direct summand Q_2 of Q such that $\alpha(\ker \nu_M|_{P_1}) \subseteq \ker \nu_N$, $\alpha|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha|_{P_2})^{-1}(\ker \nu_N|_{Q_2}) \subseteq \ker \nu_M|_{P_2}$, $M = \nu_M(P_1) \oplus \nu_M(P_2)$ and $\nu_N(Q_2) \leq_{\oplus} N$.*

The following is a consequence of Proposition 1, Theorems 2 and 5 and [8, Corollary 3.4].

Proposition 10. *Let A, B_1 and B_2 be lifting modules with the projective lifting covers. Assume that B_i is generalized B_j-projective for $i, j \in \{1, 2\}$ ($i \neq j$).*

(1) *If A is generalized B_i-projective ($i = 1, 2$), then A is generalized $B_1 \oplus B_2$-projective.*
(2) *If B_i is generalized A-projective ($i = 1, 2$), then $B_1 \oplus B_2$ is generalized A-projective.*

Finally, we give conditions for a direct sum of lifting modules with the projective lifting covers to be lifting.

Corollary 11. *Let $M_1, \ldots, M_n$ be lifting modules (resp. lifting modules with FIEP) which have the projective lifting covers and put $M = \oplus_{i=1}^{n} M_i$. Then the following conditions are equivalent:*

(a) (i) M *is lifting, and*

(ii) *the decomposition* $M = \oplus_{i=1}^{n} M_i$ *is exchangeable (resp.* M *satisfies FIEP).*

(b) M_i *is generalized* M_j*-projective for any distinct* $i, j \in \{1, \ldots, n\}$.

(c) M_i *is almost* M_j*-projective for any distinct* $i, j \in \{1, \ldots, n\}$.

Proof. By Propositions 1 and 10, Theorem 2, [8, Corollary 3.4 and Theorem 3.7] and induction. □

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