ON τ-TILTING FINITENESS OF TENSOR PRODUCT ALGEBRAS BETWEEN SIMPLY CONNECTED ALGEBRAS

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Abstract. This report is based on joint work with Qi Wang ([8]). The aim of this report is to discuss the finiteness of τ-tilting modules over the tensor product of two simply connected algebras. Moreover, we completely determine τ-tilting finite tensor products between path algebras. In addition, we determine the boundary of τ-tilting finiteness of tensor products between simply connected algebras in most cases.

1. Introduction

Throughout this paper, we will use the symbol k to denote an algebraically closed field, and tensor products are always taken over k. An algebra is always assumed to be an associative basic connected finite-dimensional k-algebra. For an algebra A, we write A^{op} for the opposite algebra of A. Modules are always finitely generated right A-modules. We denote by $\mathsf{mod}\text{-}A$ the category of modules over A. For simplicity of notation, let $\vec{A}_n$ stand for the Dynkin quiver of type A associated with the linear orientation.

Let A be an algebra. The notion of support τ-tilting A-modules was introduced in [2] as to complete the class of classical tilting modules from the viewpoint of mutations. The set of isomorphism classes of support τ-tilting modules is related to several sets of important objects arising from representation theory. For example, it is well-known that there are bijections between the set of isomorphism classes of support τ-tilting A-modules and

- the set of two-term silting complexes in the perfect derived category,
- functorially finite torsion classes in $\mathsf{mod}\text{-}A$,
- the set of left finite semibricks,
- t-structures and co-t-structures.

Therefore, the study of support τ-tilting modules has applications to those representation-theoretic classifications. In this context, τ-tilting finite algebra is introduced by Demonet, Iyama and Jasso in [6]. Such algebras are studied by several authors, for example [1], [3]. Moreover, the second author Q. Wang showed that a simply connected algebra is τ-tilting finite if and only if it is representation-finite ([9]). In the report, we focus on the τ-tilting finiteness for the tensor product $A \otimes B$ between two τ-tilting finite simply connected algebras A and B.

The detailed version of this paper will be submitted for publication elsewhere.

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2. Tensor product algebras, simply connected algebras, and τ-tilting finite algebras

2.1. **Tensor product algebras.** Let A and B be algebras. Then the tensor product $A \otimes B$ can be given the structure of a k-algebra by defining the multiplication on the elements of the form $a \otimes b$ by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. We call the algebra $A \otimes B$ *the tensor product of algebras* A and B. For example, the $n \times n$ lower triangular matrix algebra of an algebra A, that is,

$$T_n(A) = \begin{pmatrix} A & 0 & \cdots & 0 \\ A & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & A \end{pmatrix}$$

is isomorphic to $A \otimes \mathsf{k}\vec{A}_n$.

A presentation of the tensor product algebra $A \otimes B$ by a quiver and relations is given from the presentations of A and B. Assume that $A \simeq \mathsf{k}Q_A/\mathcal{I}_A$ and $B \simeq \mathsf{k}Q_B/\mathcal{I}_B$ are two bound quiver algebras. To give a presentation of $A \otimes B$, we define the tensor product of bound quivers $(Q_A, \mathcal{I}_A)$ and $(Q_B, \mathcal{I}_B)$, say $(Q_A \otimes Q_B, \mathcal{I}_A \diamond \mathcal{I}_B)$, as follows.

- The quiver $Q_A \otimes Q_B$ has the vertex set $(Q_A \otimes Q_B)_0 = (Q_A)_0 \times (Q_B)_0$ and the arrow set $(Q_A \otimes Q_B)_1 = ((Q_A)_1 \times (Q_B)_0) \cup ((Q_A)_0 \times (Q_B)_1)$ with the source map s and the target map t defined by

$$s(\alpha \times v) = s_A(\alpha) \times v, \ s(u \times \beta) = u \times s_B(\beta),$$
$$t(\alpha \times v) = t_A(\alpha) \times v, \ t(u \times \beta) = u \times t_B(\beta)$$

  for $(\alpha, v) \in (Q_A)_1 \times (Q_B)_0$ and $(u, \beta) \in (Q_A)_0 \times (Q_B)_1$, where $s_A(\alpha)$ (resp. $t_A(\alpha)$) is the source of α (resp. the target of α) and $s_B(\beta)$ (resp. $t_B(\beta)$) is the source of β (resp. the target of β).
- The ideal $\mathcal{I}_A \diamond \mathcal{I}_B$ in $\mathsf{k}(Q_A \otimes Q_B)$ is generated by $((Q_A)_0 \times \mathcal{I}_B) \cup (\mathcal{I}_A \times (Q_B)_0)$ and elements of the form $(a, \beta_{cd})(\alpha_{ab}, d) - (\alpha_{ab}, c)(b, \beta_{cd})$, where α_{ab} and β_{cd} run through all arrows $\alpha_{ab} : a \to b$ in $(Q_A)_1$ and $\beta_{cd} : c \to d$ in $(Q_B)_1$.

Then, the pair $(Q_A \otimes Q_B, \mathcal{I}_A \diamond \mathcal{I}_B)$ becomes a presentation of $A \otimes B$.

Example 1. Let A and B be the following algebras:

$$A := \mathsf{k}(1 \xrightarrow{x} 2 \xrightarrow{y} 3)/(xy), \quad B := \mathsf{k}(1' \xrightarrow{\alpha} 2' \xrightarrow{\beta} 3' \xrightarrow{\gamma} 4' \xleftarrow{\delta} 5')/(\alpha\beta\gamma).$$

Then the tensor product $A \otimes B$ is presented by the quiver

$$\begin{array}{ccccccccc}
(1,1') & \xrightarrow{\alpha_1} & (1,2') & \xrightarrow{\beta_1} & (1,3') & \xrightarrow{\gamma_1} & (1,4') & \xleftarrow{\delta_1} & (1,5') \\
{\scriptstyle x_{1'}}\downarrow & & {\scriptstyle x_{2'}}\downarrow & & {\scriptstyle x_{3'}}\downarrow & & \downarrow{\scriptstyle x_{4'}} & & \downarrow{\scriptstyle x_{5'}} \\
(2,1') & \xrightarrow{\alpha_2} & (2,2') & \xrightarrow{\beta_2} & (2,3') & \xrightarrow{\gamma_2} & (2,4') & \xleftarrow{\delta_2} & (2,5') \\
{\scriptstyle y_{1'}}\downarrow & & {\scriptstyle y_{2'}}\downarrow & & {\scriptstyle y_{3'}}\downarrow & & \downarrow{\scriptstyle y_{4'}} & & \downarrow{\scriptstyle y_{5'}} \\
(3,1') & \xrightarrow[\alpha_3]{} & (3,2') & \xrightarrow[\beta_3]{} & (3,3') & \xrightarrow[\gamma_3]{} & (3,4') & \xleftarrow[\delta_3]{} & (5,5')
\end{array}$$

and the ideal generated by $\alpha_i\beta_i\gamma_i$ $(i = 1,2,3)$, $x_{k'}y_{k'}$ $(k = 1,2,\ldots,5)$ and all commutativity relations for each square.

2.2. **Simply connected algebras.** In this subsection, we recall the definition and some properties of simply connected algebras. For details, see [5].

Let $(Q,\mathcal{I})$ be a connected bound quiver. For an arrow $\alpha \in Q_1$, we write α^{-1} for the formal inverse of α. Let a and b be vertices of Q. A walk from a to b is a formal composition $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_m^{\varepsilon_m}$, where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{\pm 1\}$ for $i = 1,2,\ldots,m$. For each vertex $a \in Q_0$, we understand the trivial path e_a as the stationary walk at a. If w is a walk from a to b and w' is a walk from b to c, the multiplication ww' is given by concatenation of w and w'. We denote by Q^* the set of all walks of Q. Then, the homotopy relation $\sim_{\mathcal{I}}$ is defined to be the smallest equivalence relation on Q^* satisfying the following three conditions.

- $\alpha\alpha^{-1} \sim_{\mathcal{I}} e_a$ and $\alpha^{-1}\alpha \sim_{\mathcal{I}} e_b$ for each arrow $a \xrightarrow{\alpha} b$.
- For each minimal relation $\sum_{i=1}^{m} \lambda_i w_i$ in $\mathcal{I}$, we have $w_i \sim_{\mathcal{I}} w_j$ for all $1 \leq i,j \leq m$.
- If u,v,w and w' are walks such that $u \sim_{\mathcal{I}} v$ and $w \sim_{\mathcal{I}} w'$, then we have $wuw' \sim_{\mathcal{I}} wvw'$ whenever the multiplications are defined.

We write $[w]$ for the equivalence class of a walk w. The multiplication on Q^* induces the multiplication $[w]\cdot[w'] = [ww']$.

Let $a \in Q_0$ be a fixed vertex, $\pi_1(Q,\mathcal{I},a)$ the set of equivalence classes of all walks from a to a. It is easily seen that $\pi_1(Q,\mathcal{I},a)$ becomes a group via the above multiplication. It is well-known that the group $\pi_1(Q,\mathcal{I},a)$ does not depend on the choice of $a \in Q_0$. We call the group $\pi_1(Q,\mathcal{I}) := \pi_1(Q,\mathcal{I},a)$ the *fundamental group* of $(Q,\mathcal{I})$.

A connected triangular algebra A is called *simply connected* if, for every presentation $(Q,\mathcal{I})$ of A, the fundamental group $\pi_1(Q,\mathcal{I})$ is trivial.

Example 2. (1) Let $A \simeq \mathsf{k}Q/\mathcal{I}$ be a bound quiver algebra such that Q is a tree. Then, A is simply connected.

(2) The quiver of a simply connected Nakayama algebra is $\vec{A}_n$ for some $n \geq 1$.

Remark 3. Let A and B be algebras. Then, $A \otimes B$ is simply connected if and only if A and B are simply connected.

2.3. **τ-tilting finite algebras.** In this subsection, we recall the definition of τ-tilting finite algebras and collect some results on τ-tilting finite algebras which are needed to discuss τ-tilting finiteness of algebras, see [2, 6]

Definition 4. Let A be an algebra, and τ the Auslander–Reiten translation on $\mathsf{mod}\text{-}A$. A module $M \in \mathsf{mod}\text{-}A$ is *τ-rigid* if $\mathsf{Hom}_A(M,\tau M) = 0$, and it is *$\tau$-tilting* if, in addition, the number of non-isomorphic indecomposable direct summands of M coincides with the number of isomorphism classes of simple A-modules. We call M *support τ-tilting* if there is an idempotent $e \in A$ such that M is a τ-tilting module over A/AeA. The algebra A is called *τ-tilting finite* if there are only finitely many isomorphism classes of basic τ-tilting A-modules.

According to [6], the following statements are equivalent for an algebra A:

- A is τ-tilting finite.
- A has only finitely many isomorphism classes of support τ-tilting modules.
- A has only finitely many isomorphism classes of A-modules X such that $\mathsf{End}_A(X)$ is a division algebra. Such a module X is called a *brick*.

Example 5. (1) All representation-finite algebras are τ-tilting finite.

(2) Any local algebra Λ has precisely two basic support τ-tilting modules Λ and 0. Thus, Λ is τ-tilting finite.

(3) Let $A = \mathsf{k}Q$, where Q is acyclic. By Gabriel's theorem, A is representation-finite if and only if Q is one of Dynkin quivers. If Q is not a Dynkin quiver, the Auslander–Reiten quiver of A contains a preprojective component which has infinitely many vertices. Since any preprojective module is a brick, A is τ-tilting infinite. As a consequence, A is τ-tilting finite if and only if Q is a Dynkin quiver.

It is well-known that if A is τ-tilting finite, then the following assertions hold ([2, 6]).

(1) The quotient algebra A/I is τ-tilting finite for any two-sided ideal I in A.
(2) The idempotent truncation eAe is τ-tilting finite for any idempotent e of A.
(3) The opposite algebra A^{op} is τ-tilting finite.

3. τ-tilting finiteness of tensor product algebras

To determine that a tensor product algebra of simply connected algebras is τ-tilting finite or not, we have the following strategy.

(a) As there are surjective k-algebra homomorphisms $A \otimes B \to A$ and $A \otimes B \to B$, it is enough to consider when A and B are τ-tilting finite.
(b) If A, B and C are non-local algebras, then $A \otimes B \otimes C$ is τ-tilting infinite [3, 8]. Thus, we only consider tensor product algebras which have exactly two components.
(c) The tensor product algebra $A \otimes B$ is τ-tilting finite if and only if $A \otimes B$ is representation-finite since $A \otimes B$ is also simply connected.
(d) A simply connected algebra is representation-finite if and only if it does not have one of concealed algebras of Euclidean type $\widetilde{D}_n$ $(n \geq 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ as a factor algebra. Such algebras are classified by Happel–Vossieck [7]. Therefore, one can determine that a simply connected algebra is τ-tilting finite or not.

3.1. The case of path algebras.

In the first place, we classify τ-tilting finite tensor products $A \otimes B$ when one of A and B is a path algebra, and this classification is complete. We denote by $\mathbb{A}_n$ $(n \geq 1)$ the Dynkin diagram of type A_n. The first main result is as follows.

Theorem 6. *Let A be a path algebra of finite connected acyclic quiver with $n \geq 2$ simple modules. Then, the following statements hold.*

(1) *Let B be a path algebra. Then, $A \otimes B$ is τ-tilting finite if and only if $A \simeq \mathsf{k}(1 \to 2)$ and B is isomorphic to one of path algebras of $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$.*
(2) *Let B be a simply connected algebra. If $\mathsf{k}(1 \to 2) \otimes B$ is τ-tilting finite, then any connected component of the separated quiver of the quiver of B is of type $\mathbb{A}_n$.*
(3) *Assume that $n \geq 3$ and B is a simply connected algebra which is not a path algebra. Then, $A \otimes B$ is τ-tilting finite if and only if A is isomorphic to a path algebra of $\mathbb{A}_3$ and B is isomorphic to a Nakayama algebra with radical square zero.*

By the above result, we have determined τ-tilting finite path algebras AQ with coefficients in a path algebra A. We remark that the statement (2) in the above result is included in [3, Theorem 3.2].

Example 7 (Method to show τ-tilting infiniteness)**.** We define $\varepsilon := (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 \in \{+, -\}$ to be the orientation of $\mathbb{A}_3$ as follows.

$$\begin{cases} i \longrightarrow i+1 & \text{if } \varepsilon_i = +, \\ i+1 \longrightarrow i & \text{if } \varepsilon_i = -. \end{cases}$$

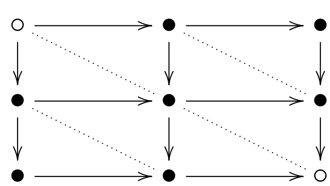
We write $\mathbf{A}_3^{\varepsilon}$ for the path algebra of type A associated with the orientation ε.

From now on, we show that the tensor product $\mathbf{A}_3^{\varepsilon} \otimes \mathbf{A}_3^{\omega}$ is τ-tilting infinite for any choice of ε and ω. Then, we need only to consider the following four cases:

- $\varepsilon = (++), \quad \omega = (++)$
- $\varepsilon = (++), \quad \omega = (-+)$
- $\varepsilon = (+-), \quad \omega = (+-)$
- $\varepsilon = (+-), \quad \omega = (-+)$

For each case, we prove that the tensor product $\mathbf{A}_3^{\varepsilon} \otimes \mathbf{A}_3^{\omega}$ has a tame concealed algebra as a quotient, which is indicated by the black points below. Here, all squares of the quiver below are commutative.

Now, we consider the case $\varepsilon = (++), \omega = (++)$. The algebra $\mathbf{A}_3^{(++)} \otimes \mathbf{A}_3^{(++)}$ is presented as follows.



Then, the algebra $\mathbf{A}_3^{(++)} \otimes \mathbf{A}_3^{(++)}$ admits a tame concealed algebra of type $\widetilde{D_4}$ as a factor, see the Happel–Vossieck list [7]. This implies that $\mathbf{A}_3^{(++)} \otimes \mathbf{A}_3^{(++)}$ is τ-tilting infinite.

Other cases can be shown in the same way.

3.2. **General cases.** Now, we discuss the τ-tilting finiteness of the tensor product of algebras $A \otimes B$ such that A and B are simply connected algebras which are not path algebras. We may assume that both A and B have at least 3 simple modules in this section. Recall that A is a simply connected Nakayama algebra (Nakayama algebra for short) if and only if the Gabriel quiver of A is of the form $\vec{A}_n$. From simply observation, we have the following.

Proposition 8. *Let A and B be two simply connected algebras. Then the following statements hold.*

1. *If both A and B are not Nakayama algebras, then $A \otimes B$ is τ-tilting infinite.*
2. *If A is a Nakayama algebra which is not radical square zero, and B is not a Nakayama algebra, then $A \otimes B$ is τ-tilting infinite.*
3. *If both A and B are Nakayama algebras which are not radical square zero, then $A \otimes B$ is τ-tilting infinite.*
4. *If both A and B are Nakayama algebras with radical square zero, then $A \otimes B$ is τ-tilting finite.*

Proof. (1), (2), and (3) We notice that there are surjections $A \otimes B \to \mathbf{A}_3^{\varepsilon} \otimes \mathbf{A}_3^{\omega}$ for some orientations ε and ω. Thus, the assertion follows from the fact that $\mathbf{A}_3^{\varepsilon} \otimes \mathbf{A}_3^{\omega}$ is τ-tilting infinite (see, Example 7).

(4) Let A and B be two simply connected Nakayama algebras with radical square zero. By the construction of a presentation of $A \otimes B$, it is special biserial. Therefore, the algebra $A \otimes B$ is of finite representation type. □

In the case that both A and B are not path algebras, we may give a visualization table below to illustrate the τ-tilting finiteness of $A \otimes B$. In the table below, F means τ-tilting finite, IF means τ-tilting infinite, and "F or IF" means that there are both cases. We denote by $\mathsf{rad}(A)$ the Jacobson radical of A and by $|A|$ the number of isomorphism classes of simple A-modules.

$A \otimes B$ (A, B: simply connected)			B :Nakayama			B:Not Nakayama		
			$\mathsf{rad}^2 = 0$		$\mathsf{rad}^2 \neq 0$			
			$n = 3$	$n \geq 4$		$\lvert B\rvert = 3$	$\lvert B\rvert = 4$	$\lvert B\rvert \geq 5$
A:Nakayama	$\mathsf{rad}^2 = 0$	$n = 3$	F	F	Open	F	F or IF	F or IF
		$n \geq 4$	F	F	F or IF	F	F or IF	IF
	$\mathsf{rad}^2 \neq 0$		Open	F or IF	IF	IF	IF	IF
A:Not Nakayama		$\lvert A\rvert = 3$	F	F	IF	IF	IF	IF
		$\lvert A\rvert = 4$	F or IF	F or IF	IF	IF	IF	IF
		$\lvert A\rvert \geq 5$	F or IF	IF	IF	IF	IF	IF

3.3. **The case that B is not Nakayama.** In this subsection, we consider the case that B is not a Nakayama algebra. Then it is only in the case that A is isomorphic to a Nakayama algebra with radical square zero that $A \otimes B$ may be τ-tilting finite. We denote by $N(n)$ the simply connected Nakayama algebra with n simple modules and radical square zero.

Theorem 9. *Let B be a simply connected not Nakayama algebra. Then the following assertions hold.*

(1) *If B has at least 5 simple modules, then $N(n)\otimes B$ is τ-tilting infinite for all $n\geq 4$.*
(2) *If B has at least 5 simple modules and $N(3)\otimes B$ is τ-tilting finite, then B or B^{op} satisfies the following conditions.*
   (a) *B or B^{op} has the algebra*

$$\mathsf{k}\left(1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4\right) \Big/ \langle\gamma\beta\rangle,$$

   as a quotient.
   (b) *B and B^{op} do not have both the algebras*

$$\mathsf{k}\left(\begin{array}{ccccc} 1 & \xrightarrow{\alpha} & 3 & \xleftarrow{\gamma} & 4 \\ & & \downarrow{\scriptstyle\beta} & & \\ & & 2 & & \end{array}\right) \Big/ \langle\alpha\beta,\gamma\beta\rangle$$

   and (4-3) as a quotient.
(3) *If B has precisely 4 simple modules, then $N(n)\otimes B$ is τ-tilting finite if and only if either B or B^{op} is isomorphic to*

$$\mathsf{k}\left(1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4\right) \Big/ \langle\gamma\beta\rangle.$$

(4) *If B has precisely 3 simple modules, then $N(n)\otimes B$ is τ-tilting finite for all $n\geq 3$.*

3.4. **The case that B is Nakayama.** Let B be a Nakayama algebra which is not radical square zero. Then, we may suppose that B has at least 4 simple modules and B is not a path algebra. In this case, determining the τ-tilting finite tensor product of algebras is complicated. However, we have a partial solution.

Theorem 10. *Let B be a Nakayama algebra which is not radical square zero. Assume that B has the algebra*

$$\Lambda = \mathsf{k}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4)/\langle\alpha\beta\gamma\rangle$$

as a quotient. Then, $N(n)\otimes B$ is τ-tilting infinite for all $n\geq 4$.

As a corollary of our classification, we determine algebras over which enveloping algebras of simply connected algebras are τ-tilting finite. Let A be an algebra. The *enveloping algebra* of A is $A^{\mathsf{e}} := A\otimes A^{\mathsf{op}}$.

Corollary 11. *Let A be a simply connected algebra. Then, the enveloping algebra A^{e} is τ-tilting finite if and only if A is a simply connected Nakayama algebra with radical square zero.*

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