LARGE TILTING OBJECTS INDUCED BY CODIMENSION FUNCTIONS AND HOMOMORPHIC IMAGES OF COHEN–MACAULAY RINGS

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Abstract. In the derived category of a commutative noetherian ring, we explicitly construct a (co)silting object associated with an sp-filtration satisfying some condition. Then we discuss when the (co)silting object is (co)tilting.

1. Introduction

This report is based on joint work with Michal Hrbek and Jan Šťovíček [4].

Throughout, let R be a commutative noetherian ring. We denote by $\mathbf{D}(R)$ the unbounded derived category of R. A map $\Phi : \mathbb{Z} \to 2^{\operatorname{Spec} R}$ is called an *sp-filtration* of $\operatorname{Spec} R$ if $\Phi(n)$ is specialization closed and $\Phi(n) \supseteq \Phi(n+1)$ for every $n \in \mathbb{Z}$. An sp-filtration Φ is called *non-degenerate* if $\bigcup_{n\in\mathbb{Z}} = \operatorname{Spec} R$ and $\bigcap_{n\in\mathbb{Z}} \Phi(n) = \emptyset$. Alonso Tarrío, Jeremías López, and Saorín [1] showed that there is a bijection between the sp-filtrations Φ of $\operatorname{Spec} R$ and the compactly generated t-structures $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$ in $\mathbf{D}(R)$. Moreover, Šťovíček and Pospíšil [6] showed that the sp-filtrations Φ of $\operatorname{Spec} R$ also bijectively correspond to the compactly generated co-t-structures $(\mathcal{X}_\Phi, \mathcal{Y}_\Phi)$ in $\mathbf{D}(R)$ and there exists a t-structure of the form $(\mathcal{Y}_\Phi, \mathcal{Z}_\Phi)$ for each Φ. If Φ is non-degenerate, then $(\mathcal{Y}_\Phi, \mathcal{Z}_\Phi)$ and $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$ are induced, respectively, by some silting object T and some cosilting object C in the sense of [5]. See [4, §2] for more details.

Although we know the existence of T and C, this fact has been shown in an abstract way. In this report, we explicitly construct a silting object and a cosilting object inducing the t-structures $(\mathcal{Y}_\Phi, \mathcal{Z}_\Phi)$ and $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$, respectively, provided that Φ is a slice sp-filtration; see Section 2. We also discuss when such (co)silting objects are (co)tilting; see Section 3.

2. Slice sp-filtrations

Let Φ be an sp-filtration of $\operatorname{Spec} R$. We call Φ a *slice* sp-filtration of $\operatorname{Spec} R$ if it is non-degenerate and

$$\dim(\Phi(n) \setminus \Phi(n+1)) \leq 0$$

for all $n \in \mathbb{Z}$. Note that the above inequality means that there is no strict inclusion of prime ideals in $\Phi(n) \setminus \Phi(n+1)$.

To each sp-filtration Φ, we assign a function $\mathsf{f}_\Phi : \operatorname{Spec} R \to \mathbb{Z} \cup \{-\infty, \infty\}$ given by

$$\mathsf{f}_\Phi(\mathfrak{p}) := \sup\{n \in \mathbb{Z} \mid \mathfrak{p} \in \Phi(n)\} + 1$$

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for each $\mathfrak{p} \in \operatorname{Spec} R$. By definition, f_Φ is an order-preserving function, that is, $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\mathsf{f}_\Phi(\mathfrak{p}) \leq \mathsf{f}_\Phi(\mathfrak{q})$. We remark that the assignment $\Phi \mapsto \mathsf{f}_\Phi$ gives rise to a bijection from the sp-filtrations of $\operatorname{Spec} R$ to the order-preserving functions $\operatorname{Spec} R \to \mathbb{Z} \cup \{-\infty, \infty\}$. Then the non-degeneracy of an sp-filtration Φ is equivalent to that the function $\mathsf{f}_\Phi : \operatorname{Spec} R \to \mathbb{Z} \cup \{-\infty, \infty\}$ restricts to a function $\operatorname{Spec} R \to \mathbb{Z}$. In fact, a non-degenerate sp-filtration Φ is a slice sp-filtration if and only if its corresponding function f_Φ is strictly increasing, that is, $\mathfrak{p} \subsetneq \mathfrak{q}$ implies $\mathsf{f}_\Phi(\mathfrak{p}) < \mathsf{f}_\Phi(\mathfrak{q})$.

Let $\mathfrak{p} \in \operatorname{Spec} R$. We denote by $\Gamma_\mathfrak{p}$ the $\mathfrak{p}$-torsion functor $\varinjlim_{n \geq 1} \operatorname{Hom}_R(R/\mathfrak{p}^n, -) : \operatorname{Mod} R \to \operatorname{Mod} R$. Moreover, we denote by $\widehat{R_\mathfrak{p}}$ the $\mathfrak{p}$-adic completion of $R_\mathfrak{p}$. Note that $\widehat{R_\mathfrak{p}}$ admits a dualizing complex $D_{\widehat{R_\mathfrak{p}}}$ such that $H^0(D_{\widehat{R_\mathfrak{p}}}) \neq 0$ and $H^i(D_{\widehat{R_\mathfrak{p}}}) = 0$ for $i < 0$.

Theorem 1. *Let Φ be a slice sp-filtration of $\operatorname{Spec} R$. In $\mathbf{D}(R)$,*

$$T_\Phi := \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R} \Sigma^{\mathsf{f}_\Phi(\mathfrak{p})} \mathbf{R}\Gamma_\mathfrak{p} R_\mathfrak{p}$$

is a silting object and

$$C_\Phi := \prod_{\mathfrak{p} \in \operatorname{Spec} R} \Sigma^{\mathsf{ht}(\mathfrak{p}) - \mathsf{f}_\Phi(\mathfrak{p})} D_{\widehat{R_\mathfrak{p}}},$$

is a cosilting object.

Theorem 1 is proved in [4, §§4–5], where it is also proved that the silting object T_Φ induces the t-structure $(\mathcal{Y}_\Phi, \mathcal{Z}_\Phi)$ and the cosilting object C_Φ induces the t-structure $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$.

The two objects T_Φ and C_Φ can be related in the following way: If E is an injective cogenerator of $\operatorname{Mod} R$, then $\mathbf{R}\operatorname{Hom}_R(T_\Phi, E)$ is a cosilting object inducing the t-structure $(\mathcal{U}_\Phi, \mathcal{V}_\Phi)$. In other words, $\mathbf{R}\operatorname{Hom}_R(T_\Phi, E)$ and C_Φ are *equivalent* as cosilting objects; see [4, §2].

3. Cohen–Macaulay homomorphic images

Our explicit construction of T_Φ and C_Φ enables us to discuss when they are tilting and cotilting, respectively, in the sense of [5]. A necessity condition is that $\mathsf{f}_\Phi : \operatorname{Spec} R \to \mathbb{Z}$ is a *codimension function*, i.e., $\mathsf{f}(\mathfrak{q}) - \mathsf{f}(\mathfrak{p}) = 1$ whenever $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\mathfrak{p}$ is maximal under $\mathfrak{q}$. Existence of a codimension function implies that R is catenary, and hence not every commutative noetherian ring can admit a codimension function. If f_Φ is a codimension function, then we call Φ a *codimension filtration* of $\operatorname{Spec} R$.

Theorem 2. *Assume that R is a homomorphic image of a Cohen–Macaulay ring of finite Krull dimension. Let Φ be a codimension filtration of $\operatorname{Spec} R$, which exists under the assumption. Then T_Φ is a tilting object and C_Φ is a cotilting object in $\mathbf{D}(R)$.*

Kawasaki [2] proved that existence of a dualizing complex D for R implies that R is a homomorphic image of a Gorenstein ring of finite Krull dimension. Since D is a cotilting object in the bounded derived category $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$, Kawasaki's result essentially characterizes homomorphic images of Gorenstein rings in terms of cotilting objects. Replacing "Gorenstein" by "Cohen–Macaulay", we suggest the next question, where we assume that R is a commutative noetherian ring of finite Krull dimension and Φ is a codimension filtration of $\operatorname{Spec} R$:

If T_Φ (resp. C_Φ) is tilting (resp. cotilting), then is R a homomorphic image of a Cohen–Macaulay ring of finite Krull dimension?

We can affirmatively answer this question when R is a local ring of Krull dimension at most 2. Our approach partly uses another work [3] of Kawasaki. See [4, §7] for more details.

References

[1] Leovigildo Alonso Tarrío, Ana Jeremías López, and Manuel Saorín, *Compactly generated t-structures on the derived category of a Noetherian ring*, J. Algebra **324** (2010), no. 3, 313–346.

[2] Takesi Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149.

[3] Takesi Kawasaki, *Finiteness of Cousin cohomologies*, Trans. Amer. Math. Soc. **360** (2008), no. 5, 2709–2739.

[4] Michal Hrbek, Tsutomu Nakamura, and Jan Šťovíček, *Tilting complexes and codimension functions over commutative noetherian rings*, in preparation.

[5] Chrysostomos Psaroudakis and Jorge Vitória, *Realisation functors in tilting theory*, Math. Z. **288** (2018), no. 3-4, 965–1028.

[6] Jan Šťovíček and David Pospíšil, *On compactly generated torsion pairs and the classification of co-t-structures for commutative noetherian rings*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6325–6361.

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