LOCALIZATION OF EXTRIANGULATED CATEGORIES

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Abstract. In this note, we introduce the notion of localizations of extriangulated categories. These localizations cover the Serre quotient, the Verdier quotient and several other localizations in the special cases and have the universality in some sense. This note is based on [5], joint work with Hiroyuki Nakaoka (Nagoya University) and Yasuaki Ogawa (Nara University of Education).

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1. Introduction

Abelian categories, exact categories and triangulated categories are the main categorical frameworks used in homological algebra. Localizations of these categories frequently appear in representation theory, for example, the Serre quotient of abelian categories [3] and the Verdier quotient of triangulated categories [8]. In recent years, the notion of extriangulated categories is introduced in [6] and unifies exact categories and triangulated categories. So far we have not seen a uniform way to formulate localizations of extriangulated categories in the literature.

In section 2, we introduce the notion of localizations of extriangulated categories. In detail, we give a sufficient condition for a set of morphisms in an extriangulated category which makes the localization of the category has a natural extriangulated structure. Note that the localization in [5] is discussed in the case of weakly extriangulated categories, which is a more general setting.

In section 3, we deal with localizations by thick subcategories. We divide these localizations into two parts by considering particular types of thick subcategories, namely biresolving subcategories and percolating subcategories. These cover the localizations arising from the following subcategories:

- biresolving subcategories in extriangulated categories fomalize:
  - thick subcategories in triangulated categories, Verdier quotient [8]
  - Hovey twin cotorsion pairs in extriangulated categories [6]
  - biresolving subcategories in exact categories [7]
- percolating subcategories in extriangulated categories formalize:
  - thick subcategories in triangulated categories, Verdier quotient [8]
  - Serre subcategories in abelian categories, Serre quotient [3]
  - two-sided admissibly percolating subcategories in exact categories [4]

The detailed version of this paper will be submitted for publication elsewhere.

We always assume that all subcategories are full, additive and closed under isomorphisms. Throughout this note, we denote by $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ an extriangulated category. See [6] for the definition. To avoid any set-theoretic problem in considering its localizations, we assume that $\mathcal{C}$ is small.

2. Localization

For a multiplicative system $\mathcal{S}$ in $\mathcal{C}$, the localization of $\mathcal{C}$ by $\mathcal{S}$ is an additive category. The aim of this section is giving a sufficient condition for $\mathcal{S}$ such that the localization naturally becomes an extriangulated category. Localizations of abelian and triangulated categories are characterized by the universality with respect to exact and triangle functors, respectively. First we give the definition of exact functors between extriangulated categories.

Definition 1. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, $(\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and $(\mathcal{C}'', \mathbb{E}'', \mathfrak{s}'')$ be extriangulated categories.

(1) ([1, Definition 2.23]) An *exact functor* $(F, \phi)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ is a pair of an additive functor $F\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\phi\colon \mathbb{E} \Rightarrow \mathbb{E}' \circ (F^{\mathrm{op}} \times F)$ which satisfies

$$\mathfrak{s}'(\phi_{C,A}(\delta)) = [F(A) \xrightarrow{F(x)} F(B) \xrightarrow{F(y)} F(C)]$$

for any $\mathfrak{s}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ in $\mathcal{C}$.

(2) If $(F, \phi)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ and $(F', \phi')\colon (\mathcal{C}', \mathbb{E}', \mathfrak{s}') \to (\mathcal{C}'', \mathbb{E}'', \mathfrak{s}'')$ are exact functors, then their composition $(F'', \phi'') = (F', \phi')\circ(F, \phi)$ is defined by $F'' = F' \circ F$ and $\phi'' = (\phi' \circ (F^{\mathrm{op}} \times F)) \cdot \phi$.

(3) Let $(F, \phi), (G, \psi)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}', \mathbb{E}', \mathfrak{s}')$ be exact functors. A *natural transformation $\eta\colon (F, \phi) \Rightarrow (G, \psi)$ of exact functors* is a natural transformation $\eta\colon F \Rightarrow G$ of additive functors, which satisfies

$$(\eta_A)_* \phi_{C,A}(\delta) = (\eta_C)^* \psi_{C,A}(\delta) \tag{2.1}$$

for any $\delta \in \mathbb{E}(C, A)$. Horizontal compositions and vertical compositions are defined by those for natural transformations of additive functors.

A usual exact functor between exact categories coincides with the above one. Similarly, a usual triangle functor between triangulated categories coincides with the above one.

In the rest of this sectoin, we denote by $\mathcal{S}$ a set of morphisms in $\mathcal{C}$ satisfying the following condition:

(M0) $\mathcal{S}$ contains all isomorphisms in $\mathcal{C}$, and is closed by compositions. Also, $\mathcal{S}$ is closed by taking finite direct sums. Namely, if $f_i\colon X_i \to Y_i$ belongs to $\mathcal{S}$ for $i = 1, 2$, then so does $f_1 \oplus f_2\colon X_1 \oplus X_2 \to Y_1 \oplus Y_2$.

First we associate a full subcategory $\mathcal{N}_{\mathcal{S}} \subseteq \mathcal{C}$ in the following way.

Definition 2. Define $\mathcal{N}_{\mathcal{S}} \subseteq \mathcal{C}$ to be the full subcategory consisting of objects $N \in \mathcal{C}$ such that both $N \to 0$ and $0 \to N$ belong to $\mathcal{S}$.

It is obvious that $\mathcal{N}_{\mathcal{S}} \subseteq \mathcal{C}$ is an additive subcategory. In the rest, we will denote the ideal quotient by $p\colon \mathcal{C} \to \overline{\mathcal{C}} = \mathcal{C}/[\mathcal{N}_{\mathcal{S}}]$, and $\overline{f}$ will denote a morphism in $\overline{\mathcal{C}}$ represented by $f \in \mathcal{C}(X, Y)$. Also, let $\overline{\mathcal{S}}$ be the closure of $p(\mathcal{S})$ with respect to compositions with isomorphisms in $\overline{\mathcal{C}}$.

Our construction of a localization of an extriangulated category factors through an ideal quotient $\overline{\mathcal{C}}$. We denote by $\widetilde{\mathcal{C}}$ the localization of $\mathcal{C}$ by $\mathcal{S}$ and by $Q\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ a localization functor. Now we give the condition for $\overline{\mathcal{S}}$ such that $\widetilde{\mathcal{C}}$ becomes an extriangulated category.

Theorem 3. *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{S}$ a set of morphisms in $\mathcal{C}$ satisfying* (M0). *Suppose that $\overline{\mathcal{S}}$ satisfies the following conditions* (MR1),(MR2),(MR3),(MR4).

(MR1) *$\overline{\mathcal{S}}$ satisfies 2-out-of-3 with respect to compositions in $\overline{\mathcal{C}}$.*
(MR2) *$\overline{\mathcal{S}}$ is a multiplicative system in $\overline{\mathcal{C}}$.*
(MR3) *Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathfrak{s}$-triangles, and let $a \in \mathcal{C}(A, A'), c \in \mathcal{C}(C, C')$ be any pair of morphisms satisfying $a_*\delta = c^*\delta'$. If $\overline{a}$ and $\overline{c}$ belong to $\overline{\mathcal{S}}$, then there exists $\mathbf{b} \in \overline{\mathcal{S}}(B, B')$ which satisfies $\mathbf{b} \circ \overline{x} = \overline{x}' \circ \overline{a}$ and $\overline{c} \circ \overline{y} = \overline{y}' \circ \mathbf{b}$.*
(MR4) *$\overline{\mathcal{M}}_{\mathsf{inf}} := \{\mathbf{v} \circ \overline{x} \circ \mathbf{u} \mid x$ is an $\mathfrak{s}$-inflation, $\mathbf{u}, \mathbf{v} \in \overline{\mathcal{S}}\}$ is closed by compositions. Dually, $\overline{\mathcal{M}}_{\mathsf{def}} := \{\mathbf{v} \circ \overline{y} \circ \mathbf{u} \mid y$ is an $\mathfrak{s}$-deflation, $\mathbf{u}, \mathbf{v} \in \overline{\mathcal{S}}\}$ is closed by compositions.*

Then the following statements holds.

1. *We obtain an extriangulated category $(\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$. See* [5] *for explicit constructions of $\widetilde{\mathbb{E}}$ and $\widetilde{\mathfrak{s}}$.*
2. *There is an exact functor $(Q, \mu)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$ satisfying the following universality:*
   - (i) *For any exact functor $(F, \phi)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ such that $F(s)$ is an isomorphism for any $s \in \mathcal{S}$, there exists a unique exact functor $(\widetilde{F}, \widetilde{\phi})\colon (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ with $(F, \phi) = (\widetilde{F}, \widetilde{\phi}) \circ (Q, \mu)$.*
   - (ii) *For any pair of exact functors $(F, \phi), (G, \psi)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ which send any $s \in \mathcal{S}$ to isomorphisms, let $(\widetilde{F}, \widetilde{\phi}), (\widetilde{G}, \widetilde{\psi})\colon (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ be the exact functors obtained in* (i). *Then for any natural transformation $\eta\colon (F, \phi) \Rightarrow (G, \psi)$ of exact functors, there is a unique natural transformation $\widetilde{\eta}\colon (\widetilde{F}, \widetilde{\phi}) \Rightarrow (\widetilde{G}, \widetilde{\psi})$ of exact functors satisfying $\eta = \widetilde{\eta} \circ (Q, \mu)$.*

From the above result, we obtain the next result which is stated in not the ideal quotient $\overline{\mathcal{C}}$, but the given category $\mathcal{C}$.

Corollary 4. *Assume that $\mathcal{S}$ satisfies* (M0) *as before, and moreover $p(\mathcal{S}) = \overline{\mathcal{S}}$. Suppose that $\mathcal{S}$ satisfies the following conditions* (M1),(M2),(M3),(M4).

(M1) *$\mathcal{S}$ satisfies 2-out-of-3 with respect to compositions in $\mathcal{C}$.*
(M2) *$\mathcal{S}$ is a multiplicative system in $\mathcal{C}$.*
(M3) *Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \overset{\delta'}{\dashrightarrow}$ be any pair of $\mathfrak{s}$-triangles, and let $a \in \mathcal{C}(A, A'), c \in \mathcal{C}(C, C')$ be any pair of morphisms satisfying $a_*\delta = c^*\delta'$. If $a, c \in \mathcal{S}$, then there exists $b \in \mathcal{S}$ which gives the following morphism of $\mathfrak{s}$-triangles.*

$$\begin{array}{ccccccc}
A & \xrightarrow{x} & B & \xrightarrow{y} & C & \overset{\delta}{\dashrightarrow} & \\
{\scriptstyle a}\downarrow & \circlearrowright & {\scriptstyle b}\downarrow & \circlearrowright & \downarrow{\scriptstyle c} & & \\
A' & \xrightarrow[x']{} & B' & \xrightarrow[y']{} & C' & \underset{\delta'}{\dashrightarrow} &
\end{array}$$

(M4) $\mathcal{M}_{\mathsf{inf}} := \{t \circ x \circ s \mid x$ *is an* $\mathfrak{s}$*-inflation,* $s, t \in \mathcal{S}\}$ *is closed under compositions. Dually,* $\mathcal{M}_{\mathsf{def}} := \{t \circ y \circ s \mid y$ *is an* $\mathfrak{s}$*-deflation,* $s, t \in \mathcal{S}\}$ *is closed under compositions.*

Then the following holds.

(1) *We obtain an extriangulated category* $(\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$.

(2) *There exists an exact functor* $(Q, \mu)\colon (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\widetilde{\mathcal{C}}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$ *which is characterized by the same universality as stated in Theorem 3* (2).

3. Localizations by thick subcategories

In this section, we introduce the localization of extriangulated categories by thick subcategories. We start by giving the definitoin of thick subcategories of extriangulated categories.

Definition 5. Let $\mathcal{N}$ be an additive subcategory of $\mathcal{C}$. We call $\mathcal{N}$ a *thick subcategory* if it is closed under taking direct summands and satisfies the 2-out-of-3 property with respect to any $\mathfrak{s}$-triangle, that is, for any $\mathfrak{s}$-triangle $A \longrightarrow B \longrightarrow C \dashrightarrow$, if two of A, B and C belong to $\mathcal{N}$, then so does the third.

A typical example of thick subcategories is the kernel of exact functors between extriangulated categories. In triangulated categories, the above definition of thick subcategories coincides with the usual one.

Definition 6. For a thick subcategory $\mathcal{N} \subseteq \mathcal{C}$, we associate the following classes of morphisms.

$$\begin{aligned}\mathcal{L} &= \{f \in \mathcal{M} \mid f \text{ is an } \mathfrak{s}\text{-inflation with } \operatorname{Cone}(f) \in \mathcal{N}\}.\\ \mathcal{R} &= \{f \in \mathcal{M} \mid f \text{ is an } \mathfrak{s}\text{-deflation with } \operatorname{CoCone}(f) \in \mathcal{N}\}.\end{aligned}$$

If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is triangulated, then we have $\mathcal{L} = \mathcal{R}$ and the localization of $\mathcal{C}$ by $\mathcal{L}$ is just the Verdier quotient by $\mathcal{N}$.

Definition 7. Let $\mathcal{N}$ be a thick subcategory of $\mathcal{C}$. We call $\mathcal{N}$ a *biresolving subcategory* if it satisfies the following condition; for any $X \in \mathcal{C}$, there are an $\mathfrak{s}$-inflation $X \to N_1$ and an $\mathfrak{s}$-deflation $N_2 \to X$ with $N_1, N_2 \in \mathcal{N}$.

In triangulated categories, biresolving subcategories coincide with thick subcategories because every morphism is both $\mathfrak{s}$-inflation and $\mathfrak{s}$-deflation in triangulated categories, see [6]. The following theorem is the first main result in this section.

Theorem 8. *Let* $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ *be an extriangulated category and* $\mathcal{N}$ *a biresolving subcategory of* $\mathcal{C}$*. Then*

(1) $\mathcal{S} = \mathcal{R} \circ \mathcal{L} = \{f \mid f = r \circ l, r \in \mathcal{R}, l \in \mathcal{L}\}$ *satisfies the assumption in Theorem 3, hence* $\widetilde{\mathcal{C}}$ *becomes an extriangulated category.*

(2) *Moreover the localization* $\widetilde{\mathcal{C}}$ *is a triangulated category.*

Next we deal with localizations by percolating subcategories.

Definition 9. Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. A thick subcategory $\mathcal{N}$ of $\mathcal{C}$ is a *percolating subcategory* if it satisfies the following condition and its dual: for any morphism $f\colon X \to N$ with $N \in \mathcal{N}$, there are an $\mathfrak{s}$-deflation $g\colon X \to N'$ and an $\mathfrak{s}$-inflation $h\colon N' \to N$ such that $f = h \circ g$ with $N' \in \mathcal{N}$.

In exact categories, percolating subcategories coincide with two-sided admissibly percolating subcategories in [4]. In triangulated categories, percolating subcategories coincide with thick subcategories because every morphism f is a composition of f and the identity morphism.

We consider the follwing condition for $\mathcal{N}$.

(P2) If $f \in \mathcal{C}(A,B)$ is a split monomorphism such that $\overline{f}$ is an isomorphism in $\overline{\mathcal{C}}$, then there exist $N \in \mathcal{N}$ and $j \in \mathcal{C}(N,B)$ such that $[f\ j]\colon A \oplus N \to B$ is an isomorphism in $\mathcal{C}$.

(P3) $\operatorname{Ker}\big(\mathcal{C}(X,A) \xrightarrow{l\circ -} \mathcal{C}(X,B)\big) \subseteq [\mathcal{N}](X,A)$ holds for any $X \in \mathcal{C}$ and any $l \in \mathcal{L}(A,B)$. Dually, $\operatorname{Ker}\big(\mathcal{C}(C,X) \xrightarrow{-\circ r} \mathcal{C}(B,X)\big) \subseteq [\mathcal{N}](C,X)$ holds for any $X \in \mathcal{C}$ and any $r \in \mathcal{R}(B,C)$.

If $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is an exact category, then every percolating subcategory of $\mathcal{C}$ always satisfies (P2) and (P3). The following theorem is the second main result in this section.

Theorem 10. *Let $(\mathcal{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category and $\mathcal{N}$ a percolating subcategory of $\mathcal{C}$. Assume that $\mathcal{N}$ satisfies* (P2) *and* (P3)*. Then $\mathcal{S} = \mathcal{L}\circ\mathcal{R} = \{f \mid f = l\circ r, r \in \mathcal{R}, l \in \mathcal{L}\}$ satisfies the assumption of Corollary 4, hence $\widetilde{\mathcal{C}}$ becomes an extriangulated category.*

This covers Theorem 8.1 in [4]. If $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is abelian, then percolating subcategories coincide with Serre subcategories, and $\mathcal{S}$ coincides with the set of all morphisms whose kernels and cokernels belong to $\mathcal{N}$. Hence the localization is just the Serre quotient. If $(\mathcal{C},\mathbb{E},\mathfrak{s})$ is triangulated, then $\mathcal{S} = \mathcal{L}$ holds, and the localization is just the Verdier quotient.

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