The 54th Symposium on Ring Theory

and Representation Theory

ABSTRACT

Saitama University, Saitama

September 6 – 9, 2022

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Combinatorics of quasi-hereditary structures

Yuta Kimura

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott [1] as a tool to study highest weight theories which arise in the representation theories of semi-simple complex Lie algebras and reductive groups. There are many examples of such algebras, Schur algebras, algebras of global dimension at most two, incidence algebras and many more.

A quasi-hereditary algebra is an Artin algebra together with a partial order on its set of isomorphism classes of simple modules which satisfies certain conditions. In the examples above, the partial order predated (and motivated) the theory, so the choice was clear (see [3]). However, there are instances of quasi-hereditary algebras having many possible choices of the partial ordering. So one may wonder about all the possible orderings. In this talk we will study such all possible choices of ordering.

We explain our results. Let A be an Artin algebra and I a set indexing the isomorphism classes of simple A-modules. For each partial order $\lhd$ on I, we have standard modules $\{\Delta(i) \mid i \in I\}$. A pair $(A, \lhd)$ is called a quasi-hereditary algebra if A is a Δ-filtered module. In this case, it is known that $(I, \lhd)$ is an "admissible" partial order [2]. Therefore, for two admissible partial orders $\lhd, \lhd'$ on I, we write $\lhd \sim \lhd'$ if $\Delta(i) \simeq \Delta'(i)$ for any $i \in I$, where Δ, Δ' are standard modules with respect to $\lhd, \lhd'$, respectively. We denote by $\mathsf{qh.str}(A)$ the equivalence classes of admissible partial orders on I by $\sim$, and call each element of $\mathsf{qh.str}(A)$ a quasi-hereditary structure.

We see that $\lhd_1 \sim \lhd_2$ if and only if characteristic tilting modules T_1 of $(A, \lhd_1)$ and T_2 of $(A, \lhd_2)$ are isomorphic. This implies that $\mathsf{qh.str}(A)$ admits a partial order induced from that of tilting modules introduced by Happel and Unger. We have the following theorem.

Theorem 1. *Let A be a path algebra of an equioriented quiver of type A_n over a field. Then there exist explicit bijections between*

(1) $\mathsf{qh.str}(A)$,
(2) *Binary trees with n vertices, and*
(3) *Isomorphism classes of tilting A-modules.*

For two finite acyclic quivers Q^1, Q^2 with sinks v^1, v^2, respectively, let Q be a quiver obtained by concatenating Q^1 and Q^2 at $v^1 = v^2$. We see that $\mathsf{qh.str}(kQ) \simeq \mathsf{qh.str}(kQ^1) \times \mathsf{qh.str}(kQ^2)$. This gives an explicit description of kQ for a Dynkin quiver Q of type A_n. Moreover, this isomorphism enables us to count up the number of quasi-hereditary structures of path algebras of Dynkin type $\mathrm{D}, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$. If time permits, we see when $\mathsf{qh.str}(kQ)$ becomes a lattice.

This talk is based on joint work with M. Flores and B. Rognerud [4].

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CONNECTEDNESS OF QUASI-HEREDITARY STRUCTURES

Yuichiro Goto

Quasi-hereditary algebras, introduced by Cline, Parshall and Scott, are generalization of hereditary algebras [1]. Moreover Dlab and Ringel showed the following theorem.

Theorem 1 ([2] Theorem 1). *Let A be a finite dimensional algebra. Then A is hereditary if and only if A is quasi-hereditary for all total orders.*

In view of the above, we consider quasi-hereditary structures, which are defined below, for a given algebra. We will use permutations instead of total orders to indicate difference of two orders.

Let K be an algebraically closed field and fix a finite dimensional K-algebra A with pairwise orthogonal primitive idempotents $e_1, \dots, e_n$. Let $\mathfrak{S}_n$ be the symmetric group on n letters and $\sigma \in \mathfrak{S}_n$. We say that σ gives a quasi-hereditary structure of A if $e_{\sigma^{-1}(1)}, \dots, e_{\sigma^{-1}(n)}$ with this order make A quasi-hereditary. We write this quasi-hereditary algebra by (A, σ) and the standard and costandard modules by $\Delta^\sigma(i)$ and $\nabla^\sigma(i)$, respectively.

For a quasi-hereditary algebra (A, σ), we say that this is ith-twistable, if the pair $(A, \sigma_i\sigma)$ is also quasi-hereditary, where $\sigma_i \in \mathfrak{S}_n$ is the adjacent transposition $(i, i+1)$. Let X be a certain property defined for elements of $\mathfrak{S}_n$. If $\sigma, \tau \in \mathfrak{S}_n$ have the property X, and if there is a decomposition

$$\tau\sigma^{-1} = \sigma_{i_l} \cdots \sigma_{i_1}$$

into adjacent transpositions where all $\sigma_{i_k} \cdots \sigma_{i_1}\sigma$ also have the property X for $1 \le k \le l$, then we say that σ and τ are connected with respect to X.

The contents of this talk are as follows. We first consider conditions on standard or costandard modules equivalent to the fact that a given quasi-hereditary algebra is ith-twistable. This is summarized below.

Theorem 2 ([3]). *Let $e \in \mathfrak{S}_n$ be the unit. A quasi-hereditary algebra (A, e) is ith-twistable if and only if one of the following conditions holds:*

($\mathcal{E}_i$): *$E_i = 0$ and $\Delta(i+1)$ has a submodule isomorphic to $\Delta(i)^{H_i}$, where $E_i = \dim \operatorname{Ext}^1_A(\Delta(i), \Delta(i+1))$ and $H_i = \dim \operatorname{Hom}_A(\Delta(i), \Delta(i+1))$.*

($\overline{\mathcal{E}_i}$): *$\overline{E_i} = 0$ and $\nabla(i+1)$ has a factor module isomorphic to $\nabla(i)^{\overline{H_i}}$ where $\overline{E_i} = \dim \operatorname{Ext}^1_A(\nabla(i+1), \nabla(i))$ and $\overline{H_i} = \dim \operatorname{Hom}_A(\nabla(i+1), \nabla(i))$.*

Moreover, if a quasi-hereditary algebra (A, e) satisfies $E_i = 0$ and $\overline{E_i} = 0$, then (A, σ_i) is also quasi-hereditary with $\Delta^{\sigma_i} = \Delta$, $\nabla^{\sigma_i} = \nabla$.

Next, we show the connectedness of permutations with respect to giving quasi-hereditary structures, and this is the main result of this talk.

Theorem 3 ([3]). *Any two permutations are connected with respect to giving quasi-hereditary structures.*

By this result, once we know that there are two permutations giving quasi-hereditary structures, then permutations in some sense lying between them give also quasi-hereditary structures.

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Approximation by interval-decomposables and interval resolutions of 2D persistence modules

Hideto Asashiba (Shizuoka Univ., KUIAS, OCAMI), **Emerson G. Escolar** (Kobe Univ.), Ken Nakashima (Okayama Univ.), Michio Yoshiwaki (OCAMI)

This talk is based on [1]. Throughout, k is a field, $\mathcal{P}$ is a finite poset, and $A := k\mathcal{P}$ is the incidence algebra (or incidence category) of $\mathcal{P}$. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules. A subset $I \subseteq \mathcal{P}$ is said to be an *interval* if it is connected and convex. For an interval I, the A-module V_I is defined by $V_I(i) = k \ \ (i \in I_0)$, $V_I(i \leq j) = 1_k$ and 0 otherwise, is called an *interval module* over A. An A-module M is called *interval-decomposable* if M is isomorphic to a finite direct sum of interval modules. We denote by $\mathbb{I}$ and $\mathcal{I}$ the set of all intervals and the set of all interval-decompoable modules, respectively.

In topological data analysis, the interval(-decomposable) modules play a central role in one-parameter persistent homology, as they are used to express the "birth" and "death" of topological features. In case that $\mathcal{P} = (\{1, \ldots, m\}, \leq) \times (\{1, \ldots, n\}, \leq)$, an $m \times n$ commutative 2D grid, A-modules are called 2D *persistence modules*, and can be used to study the evolution of topological features varying across two parameters. We are interested in approximating 2D persistence modules using interval-decomposables.

For any finite poset $\mathcal{P}$ and any $k\mathcal{P}$-module M, we can consider homological approximations relative to intervals as follows [2]. Let $\cdots \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$ be a minimal $\mathcal{I}$-resolution of M. Analogous to the global dimension, the *interval global dimension* $\operatorname{int-gldim} A$ of $A = k\mathcal{P}$ is defined. We show the following results that hold in general, not just for the 2D commutative grid.

Proposition 1. *For any finite poset $\mathcal{P}$,* $\operatorname{int-gldim} k\mathcal{P} < \infty$.

Proposition 2. $\operatorname{int-gldim} k\mathcal{P} = \max_{I \in \mathbb{I}} \operatorname{int-dim}(\tau V_I)$, *where τ is the Auslander–Reiten translation.*

We use Proposition 2 in computational experiments, and obtain some conjectures about the value of int-gldim for the 2D commutative grids.

Next, we recall another notion of approximation using intervals. The *compressed multiplicity* is defined as $c_M(I) := d_{R_I(M)}(R_I(V_I))$, the multiplicity of $R_I(V_I)$ as a direct summand of $R_I(M)$, where R_I is some restriction functor defined using I. The following relates the compressed multiplicities to the multiplicities.

Proposition 3. *Let M be interval-decomposable. For each $I \in \mathbb{I}$, $d_M(V_I) = \sum_{S \subseteq \operatorname{cov}(I)} (-1)^{\#S} c_M(\bigvee S)$.*

For A-modules M not necessarily interval-decomposable, by regarding the right hand side (denoted by $\delta_M(V_I)$), which may be negative, as multiplicities in the split Grothendieck group of $\operatorname{mod} A$, we obtain a notion of approximation of M by interval-decomposables.

In a previous work, R_I was defined using an inclusion of the source and sink vertices of I. However, the resulting invariant does not seem to have good homological properties. Here, in the $m \times 2$ commutative grid case, we use a modified version of R_I and obtain the following linking the two notions of approximation.

Proposition 4. *Let $\mathcal{P}$ be the $m \times 2$ commutative grid, and let each X_i in an $\mathcal{I}$-resolution $0 \to X_\ell \xrightarrow{f_\ell} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$ be decomposed as $X_i \cong \bigoplus_{I \in \mathbb{I}} V_I^{r_I^{(i)}(M)}$. Then $\delta_M(V_I) = \sum_{i=0}^{\ell} (-1)^i r_I^{(i)}(M)$.*

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Characterization of 4-dimensional non-thick irreducible representations

Kazunori Nakamoto and Yasuhiro Omoda

In this talk, we deal with characterization of 4-dimensional non-thick irreducible representations of an arbitrary group. Let V be a finite-dimensional vector space over a field k.

Definition 1 (*cf.* [1] and [2])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. We say that ρ is *m-thick* if for any subspaces V_1, V_2 of V with $\dim_k V_1 = m$ and $\dim_k V_2 = \dim_k V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$. If ρ is m-thick for any $0 < m < \dim_k V$, then we say that ρ is *thick*.

Roughly speaking, m-thick representations $\rho : G \to \mathrm{GL}(V)$ have enough transitivity of the group action of G on the set of m-dimensional vector subspaces of V. In [1], we obtained several results on thick representations. Irreducibility is equivalent to 1-thickness. Any thick representations are irreducible. For a representation $\rho : G \to \mathrm{GL}(V)$ with $\dim_k V \leq 3$, ρ is thick if and only if it is irreducible. If $\dim_k V \geq 4$, then there exist many examples of non-thick irreducible representations. In [2], we have classified the (finite-dimensional) thick representations of complex connected simple Lie groups over $\mathbb{C}$.

Here let us give an example of a non-thick irreducible representation. We denote by S_n the symmetric group of degree n. We regard $\mathrm{GL}(V)^n$ and S_n as subgroups of $\mathrm{GL}(V^{\oplus n})$ by $(A_1, \ldots, A_n) \cdot (v_1, \ldots, v_n) = (A_1 v_1, \ldots, A_n v_n)$ and $\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$ for $(A_1, \ldots, A_n) \in \mathrm{GL}(V)^n, \sigma \in S_n$ and $(v_1, \ldots, v_n) \in V^{\oplus n}$, respectively. Then the semidirect product $\mathrm{GL}(V)^n \rtimes S_n$ is defined as a subgroup of $\mathrm{GL}(V^{\oplus n})$. The inclusion $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ gives a representation of $\mathrm{GL}(V)^n \rtimes S_n$.

Theorem 2. *Let $n \geq 2$ and $\dim_k V \geq 2$. The representation $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ is a non-thick irreducible representation. More precisely, $\rho_{V,n}$ is neither n-thick nor $\dim_k V$-thick.*

The following theorem gives us another way of constructing non-thick representations.

Theorem 3. *Let $\tau_1 : G \to \mathrm{GL}(V_1)$ and $\tau_2 : G \to \mathrm{GL}(V_2)$ be finite-dimensional representations of a group G over a field k. If $\dim_k V_1 \geq 2$ and $\dim_k V_2 \geq 2$, then $\tau_1 \otimes \tau_2 : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ is not 2-thick. In particular, $\tau_1 \otimes \tau_2$ is not thick.*

Let us consider 4-dimensional non-thick irreducible representations. Our main theorem is the following:

Theorem 4. *Let $\rho : G \to \mathrm{GL}(V)$ be a 4-dimensional non-thick irreducible representation of a group G. Then ρ is equivalent to one of the following two cases:*

(1) *the composition $G \xrightarrow{\phi} \mathrm{GL}(W)^2 \rtimes S_2 \xrightarrow{\rho_{W,2}} \mathrm{GL}(W \oplus W)$ with $\dim_k W = 2$, where $\phi : G \to \mathrm{GL}(W)^2 \rtimes S_2$ is a group homomorphism*

(2) *a representation $\rho' : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ with $\dim_k V_1 = \dim_k V_2 = 2$ which is equivalent to $\tau_1 \otimes \tau_2$ as projective representations, where $\tau_i : G \to \mathrm{PGL}(V_i)$ is a projective representation for $i = 1, 2$.*

The detail will be given in [3].

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Relative stable equivalences of Morita type for the principal blocks of finite groups

Naoko Kunugi and Kyoichi Suzuki

Broué[1] introduced the notion of *stable equivalences of Morita type* for blocks of finite groups, that is, stable equivalences induced by bimodules which are projective as left and as right modules. He showed that we can construct a stable equivalence of Morita type between the principal (p-)blocks of two finite groups having a common Sylow p-subgroup by gluing Morita equivalences for the principal blocks of the centralizers of all the nontrivial p-subgroups. Linckelmann[3] investigated properties of stable equivalences of Morita type and gave an equivalent condition for a stable equivalence of Morita type to be in fact a Morita equivalence. When we want to construct a Morita equivalence between two principal blocks of finite groups, we may construct a stable equivalence of Morita type by using Broue's gluing method and lift it to a Morita equivalence by using Licnkelmann's result. In fact, Morita equivalences has been confirmed in some cases in this way, for example see [4] and [2].

We consider the principal blocks of two finite groups having a common Sylow p-subgroup with a nontrivial central p-subgroup Z. In this case, we cannot use Broué's gluing method, and hence we may not construct a stable equivalence of Morita type. On the other hand, Wang-Zhang[5] introduced the notion of *relative stable equivalences of Morita type* for blocks of finite groups, which is a generalization of stable equivalences of Morita type. In this talk, we investigate stable equivalences of Morita type relative to the group $Z \times Z$ for the principal blocks.

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Tilting complexes over blocks covering cyclic blocks

Yuta Kozakai

Let p be a prime number, k an algebraically field of characteristic p, $\tilde{G}$ a finite group, and G a normal subgroup of $\tilde{G}$ having a p-power index in $\tilde{G}$. Moreover let B be a block of kG and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. For $\Lambda \in \{B, \tilde{B}\}$, we denote by $2\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic 2-term tilting complexes over Λ, and by $\text{tilt}\,\Lambda$ the set of isomorphism classes of basic tilting complexes over Λ. In [1], it is shown that the set $\text{tilt}\,\Lambda$ has a structure of partially ordered set, and the classification of tilting complexes by using this partial order is one of the themes of the representation theory of finite dimensional algebras.

On the other hand, in [2] R. Koshio and the author proved that if B satisfies the following conditions, then the induction functor $\text{Ind}_G^{\tilde{G}}(-) := k\tilde{G} \otimes_{kG} - : K^b(\text{proj } B) \to K^b(\text{proj } \tilde{B})$ induces an isomorphism between $2\text{-tilt}\,B$ and $2\text{-tilt}\,\tilde{B}$:

- any indecomposable B-module is $I_{\tilde{G}}(B)$-invariant,
- the block B is τ-tilting finite.

We can naturally expect that the induction functor $\text{Ind}_G^{\tilde{G}}(-) := k\tilde{G} \otimes_{kG} - : K^b(\text{proj } B) \to K^b(\text{proj } \tilde{B})$ induces an isomorphism between $\text{tilt}\,B$ and $\text{tilt}\,\tilde{B}$ as partially ordered sets under some assumptions. In this talk, we give a condition for the functor to induce the isomorphism between $\text{tilt}\,B$ and $\text{tilt}\,\tilde{B}$. Moreover, we focus on the case of B having a cyclic defect group.

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On IE-closed subcategories

Arashi Sakai

Given a finite dimensional algebra Λ, it is fundamental to study subcategories of the category $\mathsf{mod}\Lambda$ of finitely generated right Λ-modules. We consider *IE-closed subcategories*, subcategories closed under taking Images and Extensions. The class of IE-closed subcategories is very large as it contains all *torsion classes, torsion-free classes, wide subcategories*, and *ICE-closed subcategories* introduced in [3]. The aim of this talk is giving two results concerning IE-closed subcategories.

The notion of torsion classes is one of the most important objects in the representation theory of finite dimensional algebras. In [1], it is shown that functorially finite torsion classes are classified by *support τ-tilting modules*, a generalization of tilting modules. In [2], the concept of *τ-tilting finite algebras* is introduced. We say that a finite dimensional algebra Λ is τ-tilting finite if the set of functorially finite torsion classes in $\mathsf{mod}\Lambda$ is a finite set. It is equivalent to the condition that every torsion class in $\mathsf{mod}\Lambda$ is functorially finite. The first result is the characterization of τ-tilting finiteness by using IE-closed subcategories:

Theorem 1. [4] *Let Λ be a finite dimensional algebra. The following are equivalent.*

(1) Λ *is τ-tilting finite.*
(2) *The set of IE-closed subcategories of* $\mathsf{mod}\Lambda$ *is a finite set.*
(3) *Every IE-closed subcategory of* $\mathsf{mod}\Lambda$ *is functorially finite.*

We focus on the classification of functorially finite IE-closed subcategories. The classification result in [1] of functorially finite torsion classes is obtained from taking Ext-projectives of a given functorially finite torsion class. Since IE-closed subcategories can not be recovered from only Ext-projectives, we consider also Ext-injectives. We introduce *twin rigid modules*, a pair of rigid (*i.e.* any self-extensions are vanished) modules satisfying some conditions, and obtain the second result. It is obtained from taking Ext-projectives and Ext-injectives of a given functorially finite IE-closed subcategory under the assumption that Λ is hereditary.

Theorem 2. [4] *Let Λ be a finite dimensional hereditary algebra. Then there exist bijective correspondences between the set of functorially finite IE-closed subcategories of* $\mathsf{mod}\Lambda$ *and the set of isomorphism classes of basic twin rigid Λ-modules.*

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On τ-tilting finiteness of group algebras

Takuma Aihara and Taro Sakurai

τ-tilting theory, which deals with problems of support τ-tilting modules, was introduced by Adachi–Iyama–Reiten as a generalization of tilting theory [1]. The greatest advantage of the theory is to be compatible with mutation theory. Moreover, the theory has connections with many topics of mathematics.

In this talk, we discuss τ-tilting theory for group algebras. In particular, one explores when a group algebra is τ-tilting finite (i.e., there exist only finitely many support τ-tilting modules). We also consider influence of τ-tilting finiteness on group algebras.

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Symmetric cohomology and symmetric Hochschild cohomology of cocommutative Hopf algebras

Ayako Itaba, Yuta Shiba and Katsunori Sanada

Let G be a group and X a G-module. Motivated by topological geometry, Staic [3] defined the symmetric cohomology $\mathrm{HS}^\bullet(G, X)$ of a group G by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard resolution $\mathrm{C}^\bullet(G, X)$ which gives the group cohomology $\mathrm{H}^\bullet(G, X)$. In [4], Staic proved that the secondary cohomology group $\mathrm{HS}^2(G, X)$ is corresponding to extensions of groups which satisfies some conditions. Moreover, Staic studied the injectivity of the canonical map $\mathrm{HS}^\bullet(G, X) \longrightarrow \mathrm{H}^\bullet(G, X)$ induced by the inclusion $\mathrm{CS}^\bullet(G, X) \hookrightarrow \mathrm{C}^\bullet(G, X)$. Recently, Coconet-Todea [1] defined the symmetric Hochschild cohomology of twisted group algebras which is a generalization of group algebras.

This talk is about the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras as another generalization of group algebras. This talk is based on [2]. Let k be a field and A a cocommutative Hopf algebra over k. First, we define the symmetric cohomology $\mathrm{HS}^\bullet(A, M)$ of A with coefficients in any left A-module M by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard non-homogeneous complex $\mathrm{C}^\bullet(A, M)$ which gives the Hopf algebra cohomology $\mathrm{H}^\bullet(A, M)$. Similarly, we define the symmetric Hochschild cohomology $\mathrm{HHS}^\bullet(A, M)$ of A with coefficients in any A-bimodule M by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard non-homogeneous complex $\mathrm{C}_e^\bullet(A, M)$ which gives the Hochschild cohomology $\mathrm{HH}^\bullet(A, M)$. Next, we give an isomorphism between symmetric cohomology and symmetric Hochschild cohomology, which is a symmetric version of the classical result about cohomology of groups by Eilenberg-MacLane and cohomology of Hopf algebras by Ginzburg-Kumar.

Theorem 1 ([2, Theorem 4.5]). *Let A be a cocommutative Hopf algebra and M an A-bimodule. Then, for each $n \geq 0$, there is an isomorphism $\mathrm{HHS}^n(A, M) \cong \mathrm{HS}^n(A, {}^{\mathrm{ad}}M)$ as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action.*

Moreover, to consider the condition that symmetric cohomology coincides with classical cohomology, we investigate the projectivity of a resolution which gives symmetric cohomology.

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Characterization of eventually periodic modules and its applications

Satoshi Usui

This talk is based on [3]. The *singularity category* $\mathcal{D}_{\mathrm{sg}}(R)$, introduced by Buchweitz [1], of a left Noetherian ring R is defined to be the Verdier quotient $\mathcal{D}^{\mathrm{b}}(R\text{-mod})/\mathrm{perf}(R)$ of the bounded derived category $\mathcal{D}^{\mathrm{b}}(R\text{-mod})$ of R-modules by the full subcategory $\mathrm{perf}(R)$ consisting of complexes that are quasi-isomorphic to bounded complexes of projective R-modules. It is well known that an R-module M has finite projective dimension if and only if $M \cong 0$ in $\mathcal{D}_{\mathrm{sg}}(R)$, where we consider M as the stalk complex M concentrated in degree 0. From this point of view, studying homological properties of R-modules with infinite projective dimension captured by $\mathcal{D}_{\mathrm{sg}}(R)$ is natural.

Recall that a module M over a left artin ring R is called *eventually periodic* if its minimal projective resolution has infinite length and eventually becomes periodic. Recently, it was proved by the speaker [2] that, when R is a finite dimensional Gorenstein algebra, an R-module M is eventually periodic if and only if the Tate cohomology ring $\widehat{\mathrm{Ext}}_R^{\bullet}(M, M)$ has a non-zero invertible homogeneous element of positive degree. Here, $\widehat{\mathrm{Ext}}_R^{\bullet}(M, M)$ stands for the graded ring $\bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{Ext}}_R^{i}(M, M)$ with $\widehat{\mathrm{Ext}}_R^{i}(M, M) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(R)}(M, M[i])$ given by Yoneda product.

In this talk, we first extend the above result to the case of left artin rings. Then, as applications, we show that eventual periodicity of finite dimensional algebras is preserved under singular equivalence of Morita type with level, introduced by Wang [4]. Here, a finite dimensional algebra is called *eventually periodic* if it is eventually periodic as a bimodule over itself. Moreover, we give a necessary and sufficient condition for a finite dimensional connected Nakayama algebra to be eventually periodic.

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The spectrum of Grothendieck monoid: a new approach to classify Serre subcategories

Shunya Saito

Classifying nice subcategories of an abelian category or a triangulated category is quite an active subject which has been studied in the representation theory of algebras. A typical and classical example is Gabriel's classification of Serre subcategories. For a commutative noetherian ring R, Gabriel [1] established the bijection between Serre subcategories of the category $\mathsf{mod}\, R$ of finitely generated R-modules and specialization-closed subsets of the spectrum $\operatorname{Spec} R$ of R. So far, the classification of Serre subcategories has focused on the case of abelian categories such as the above example. In this talk, I will propose a strategy to classify Serre subcategories of an *exact category* by using the *Grothendieck monoid*.

The Grothendieck monoid $\mathsf{M}(\mathcal{E})$ is a monoid version of the Grothendieck group, which is defined for each exact category $\mathcal{E}$. The following theorem is the starting point of this talk.

Theorem 1. *For an exact category $\mathcal{E}$, there are bijections between the following sets:*

(1) *The set of Serre subcategories of $\mathcal{E}$.*
(2) *The set* $\operatorname{Face}(\mathsf{M}(\mathcal{E}))$ *of faces of the Grothendieck monoid* $\mathsf{M}(\mathcal{E})$.
(3) *The set* $\operatorname{MSpec} \mathsf{M}(\mathcal{E})$ *of prime ideals of the Grothendieck monoid* $\mathsf{M}(\mathcal{E})$.

Here a non-empty subset F of a commutative monoid M is a face *if for all $x, y \in M$, we have that $x + y \in F$ if and only if $x, y \in F$. A subset $\mathfrak{p}$ of M is a* prime ideal *if $M \setminus \mathfrak{p}$ is a face of M.*

The second set $\operatorname{Face}(\mathsf{M}(\mathcal{E}))$ can be computed purely algebraically, and its computation is much easier than examining the whole structure of the exact category $\mathcal{E}$. I will give concrete examples of classifying Serre subcategories of exact categories related to a finite dimensional algebra by using this result.

The third set $\operatorname{MSpec} \mathsf{M}(\mathcal{E})$ has a topology, which is a natural analogue of the Zariski topology on the spectrum $\operatorname{Spec} R$ of a commutative ring R. This observation gives the following theorem.

Theorem 2. *Consider the following conditions for commutative noetherian rings R and S.*

(1) $R \cong S$ *as rings.*
(2) $\mathsf{M}(\mathsf{mod}\, R) \cong \mathsf{M}(\mathsf{mod}\, S)$ *as monoids.*
(3) $\operatorname{MSpec} \mathsf{M}(\mathsf{mod}\, R) \cong \operatorname{MSpec} \mathsf{M}(\mathsf{mod}\, S)$ *as topological spaces.*
(4) $\operatorname{Spec} R \cong \operatorname{Spec} S$ *as topological spaces.*

Then the implications "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4)" hold.

It is surprisingly enough because the Grothendieck monoid $\mathsf{M}(\mathsf{mod}\, R)$ loses a lot of information and the Grothendieck group $\mathsf{K}_0(\mathsf{mod}\, R)$ never recovers the topology of $\operatorname{Spec} R$.

This talk is based on a preprint [2]

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**Noncommutative conics
in Calabi-Yau quantum projective planes I**

Haigang Hu, Masaki Matsuno and Izuru Mori

Throughout this talk, we fix an algebraically closed field k of characteristic 0. By Sylvester's theorem, it is elementary to classify commutative quadric hypersurfaces in $\mathbb{P}^{d-1}$, namely, every quadric hypersurface in $\mathbb{P}^{d-1}$ is isomorphic to $\operatorname{Proj} k[x_1, \cdots, x_d]/(x_1^2 + \cdots + x_j^2)$ for some $j = 1, \cdots, d$. The ultimate goal of our project is to classify noncommutative quadric hypersurfaces in quantum $\mathbb{P}^{d-1}$.

Definition 1. A d-dimensional quantum polynomial algebra is a noetherian connected graded algebra S such that

(1) $\operatorname{gldim} S = d < \infty$,
(2) $\operatorname{Ext}^q_S(k, S) = 0$ if $q \neq d$, and $\operatorname{Ext}^d_S(k, S) \cong k$, and
(3) $H_S(t) = 1/(1-t)^d$.

A d-dimensional quantum polynomial algebra S is a noncommutative analogue of the commutative polynomial algebra $k[x_1, \cdots, x_d]$, so the noncommutative projective scheme $\operatorname{Proj_{nc}} S$ associated to S in the sense of [1] is regarded as a quantum $\mathbb{P}^{d-1}$. It is reasonable to define a noncommutative quadric hypersurface in a quantum $\mathbb{P}^{d-1}$ as $\operatorname{Proj_{nc}} S/(f)$ where $f \in S_2$ is a regular central element.

In this talk, we will study in the case $d = 3$. For every 3-dimensional quantum polynomial algebra S, there exists a 3-dimensional Calabi-Yau quantum polynomial algebra S' such that $\operatorname{Proj_{nc}} S \cong \operatorname{Proj_{nc}} S'$, so we may restrict ourselves to the Calabi-Yau case in this talk. The aim of this talk is to classify the homogeneous coordinate algebras $A = S/(f)$.

Since there are infinitely many isomorphism classes of 3-dimensional Calabi-Yau quantum polynomial algebras S up to isomorphism of graded algebras, classification of $A = S/(f)$ is highly nontrivial. The following is the main result of this talk.

Theorem 2 ([2])**.** *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$.*

(1) *If A is commutative, then A is isomorphic to one of the following algebras:*
$$k[x, y, z]/(x^2), \;\; k[x, y, z]/(x^2 + y^2), \;\; k[x, y, z]/(x^2 + y^2 + z^2).$$
(2) *If A is not commutative, then A is isomorphic to $S^{(a,b,c)}/(\alpha x^2 + \beta y^2 + \gamma z^2)$ where*
$$S^{(a,b,c)} := k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2)$$
for some $(a, b, c) \in \mathbb{A}^3, (\alpha, \beta, \gamma) \in \mathbb{P}^2$.

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Noncommutative conics
in Calabi-Yau quantum projective planes II

Haigang Hu, Masaki Matsuno, Izuru Mori

This is a continuation of M. Matsuno's talk. Throughout this talk, let k be an algebraically closed field of characteristic 0. In this talk, we explain how we classify noncommutative conics in Calabi-Yau quantum projective planes, and give the complete classification result of these noncommutative conics. We begin by repeating the following definition.

Definition 1. A *noncommutative quadric hypersurface in a (Calabi-Yau) quantum* $\mathbb{P}^{d-1}$ is the noncommutative projective scheme $\operatorname{Proj}_{\rm nc} S/(f)$ for some d-dimensional (Calabi-Yau) quantum polynomial algebra S and for some regular central element $f \in Z(S)_2$. In particular, when $d = 3$, we say that $\operatorname{Proj}_{\rm nc} S/(f)$ is a *noncommutative conic*.

For the homogeneous coordinate algebra $A = S/(f)$ of a noncommutative quadric hypersurface, there is a unique regular central element $f^! \in Z(A^!)_2$ such that $S^! = A^!/(f^!)$. We define $C(A) := A^![(f^!)^{-1}]_0$. Smith and Van den Bergh showed that $\underline{\rm CM}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\operatorname{mod} C(A))$ in [2]. Thought it is not clear whether $A \cong A'$ implies that $C(A) \cong C(A')$ from the definition, we have the following result.

Theorem 2. [1] *Let S, S' be 3-dimensional Calabi-Yau quantum polynomial algebras,* $0 \neq f \in Z(S)_2, 0 \neq f' \in Z(S')_2$*, and* $A = S/(f), A' = S'/(f')$*. Then*

$$A \cong A' \Rightarrow \operatorname{Proj}_{\rm nc} A \cong \operatorname{Proj}_{\rm nc} A' \Rightarrow C(A) \cong C(A').$$

The calculation of $C(A)$ associated to a noncommutative conic is hard, actually we failed to calculate it directly. However, we find that we can determine $C(A)$ using geometric method, calculating the *point variety* E_A of A, and we will show that there are 9 isomorphism classes of $C(A)$.

List of $C(A)$					
$M_2(k)$,		$k\langle u,v\rangle/(uv+vu, u^2-1, v^2)$,		$k\langle u,v\rangle/(uv+vu, u^2, v^2)$,	
k^4,	$k[u]/(u^2)\times k^2$,	$(k[u]/(u^2))^{\times 2}$,	$k[u]/(u^3)\times k$,	$k[u]/(u^4)$,	$k[u,v]/(u^2,v^2)$.

We also show that there are at most 9 isomorphism classes of A above by the help of classification of (commutative) *pencils of conics*. Combining with Theorem 2, we have the following main result.

Theorem 3. [1] *There are 9 isomorphism classes of noncommutative conics* $\operatorname{Proj}_{\rm nc} A$ *in Calabi-Yau quantum* $\mathbb{P}^2$*'s.*

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Twisted Segre products and noncommutative quadric surfaces

Kenta Ueyama

The Segre product of graded algebras plays an important role in algebraic geometry and commutative algebra. The main reason is that the Segre product gives a homogeneous coordinate ring of the product of projective varieties. In this talk, with the aim of contributing to the further development of noncommutative algebraic geometry, we introduce the notion of the twisted Segre product $A \circ_\psi B$ of (not necessarily commutative) $\mathbb{Z}$-graded algebras A and B with respect to a twisting map $\psi : B \otimes A \to A \otimes B$. Note that if ψ is the flip map, i.e., $\psi(b \otimes a) = a \otimes b$ for all $a \in A$ and $b \in B$, then $A \circ_\psi B$ coincides with the usual Segre product of A and B.

We first discuss the following theorem.

Theorem 1. *Let A and B be noetherian Koszul AS-regular algebras, and let $\psi : B \otimes A \to A \otimes B$ be a twisting map. Assume that the twisted Segre product $A \circ_\psi B$ is noetherian. Then the noncommutative projective scheme $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension.*

Since noetherian Koszul AS-regular algebras are noncommutative analogues of standard graded polynomial rings, the above theorem can be regarded as a noncommutative analogue of the fact that $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is smooth.

We then discuss the following theorem.

Theorem 2. *Let $A = k[u, v], B = k[x, y]$ be standard graded polynomial rings in two variables, and let $\psi : B \otimes A \to A \otimes B$ be a twisting map. Assume that ψ is diagonal. Then the following statements hold.*

(1) *The twisted Segre product $A \circ_\psi B$ is a noncommutative quadric surface, i.e., there exist a 4-dimensional noetherian Koszul AS-regular algebra S with Hilbert series $H_S(t) = (1-t)^{-4}$ and a regular normal homogeneous element $f \in S$ of degree 2 such that $A \circ_\psi B \cong S/(f)$. In particular, $A \circ_\psi B$ is a 3-dimensional noetherian Koszul AS-Gorenstein algebra and $\mathsf{qgr}\, A \circ_\psi B$ has finite global dimension.*

(2) *There exists an equivalence of triangulated categories*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k \times k),$$

where $\underline{\mathsf{CM}}^{\mathbb{Z}}(A \circ_\psi B)$ is the stable category of graded maximal Cohen-Macaulay modules over $A \circ_\psi B$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, k \times k)$ is the bounded derived category of finite dimensional modules over $k \times k$.

It turns out that $A \circ_\psi B$ appearing in the above theorem has a nice property similar to the usual Segre product $A \circ B \cong k[X, Y, Z, W]/(XW - YZ)$, which is the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$.

This talk is based on joint work [1] with Ji-Wei He (Hangzhou Normal University).

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On Generalized Nakayama-Azumaya's Lemma

Masahisa SATO

Before we talk a topic of the title, we would like to announce formally that the Ware problem is affirmative [3] as it was reported in [4]. (cf. It had been reported that this problem was negative in [2].)

Let R be an associative ring and $J(R)$ its Jacobson radical. The following Nakayama-Azumaya's Lemma is well known.

Nakayama-Azumaya's Lemma *Let M be a finitely generated right R-module. If M satisfies $MJ(R) = M$, then $M = 0$.*

Our main theorem is a generalization of this lemma.

Generalized Nakayama-Azumaya's Lemma *Let M be a direct summand of a direct sum of finitely generated right R-modules. If M satisfies $MJ(R) = M$, then $M = 0$.*

In the case M is a projective module, Nakayama-Azumaya's Lemma holds by [1]. This is the special case for the above result by taking R for all finitely generated modules.
In this sense, Generalized Nakayama-Azumaya's Lemma unifies the above two known results for finitely generated modules and projective modules.

Generalized Nakayama-Azumaya's Lemma is shown by the following two-step investigation.
First step is to show the following equivalent properties [5].

Proposition *Generalized Nakayama-Azumaya's Lemma holds if and only if there are no non-zero right module M (called NAS-module) which satisfies the following properties.*

(1) *There is an ascending chain $M_1 \subset M_2 \subset \cdots$ of finitely generated submodules of M such that $M_i \subset M_{i+1}J(R)$ for any $i \in \mathbb{N}$ and $M = \bigcup_{i\in\mathbb{N}} M_i$. Consequently, $M = MJ(R)$.*

(2) *Let $f : \bigoplus_{i\in\mathbb{N}} M_i \to M$ be the homomorphism given by $f((x_i)) = \sum_{i\in\mathbb{N}} x_i$ for any $(x_i) \in \bigoplus_{i\in\mathbb{N}} M_i$. Then there is a homomorphism $g : M \to \bigoplus_{i\in\mathbb{N}} M_i$ such that $fg = 1_M$.*

(3) *$M_i \cap g(M) = M_i \cap \ker f = 0$ for any $i \in \mathbb{N}$.*

Next step is to show the non-existence of a module (called WNAS-module) which satisfies the properties (1) and (2) [6].

As one of applications of Generalized Nakayama-Azumaya's Lemma, we can show the existence of maximal submodules of some kinds of module M [6].

Theorem *Let M be a non-zero direct summand of a direct sum of finitely generated right R-modules, then M has a maximal submodule.*

Remark In the above theorem, the assumption that M is a direct summand is necessary. In fact, there is a commutative uniserial ring R with $J(R)^2 = J(R) \neq 0$ such that its unique maximal ideal $J(R)$ has no maximal submodules [4].

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A new framework of partially additive algebraic geometry

Shingo Okuyama

Introduction. In his 1957 paper[1], J.Tits observed that the correspondence of geometries over a field k and the Chevalley groups over k developed in that paper specializes, when k is the hypothetical "field of characteristic one", to the correspondence of finite complexes and the Weyl groups of those Chevalley groups. In the early 1990s, Manin, based on the 'beautiful ideas of Deninger and Kurokawa', proposed to use this hypothetical field of characteristic one to solve the Riemann hypothesis. Since then, there have been many attempts to establish a foundation for these ideas, but it seems that the project has not been settled down yet.

In this talk, we develop an elementary theory of partially additive rings as a foundation of $\mathbb{F}_1$-geometry. Our approach is so concrete that an analog of classical algebraic geometry is established very straightforwardly. As applications, (1) we construct a kind of affine group scheme $\mathbb{GL}_n$ whose value at a commutative ring R is the group of $n \times n$ invertible matrices over R and at $\mathbb{F}_1$ is the n-th symmetric group, and (2) we construct a projective space $\mathbb{P}^n$ as a kind of scheme and count the number of points of $\mathbb{P}^n(\mathbb{F}_q)$ for $q = 1$ or $q = p^n$ a power of a rational prime. Concerning the existing theories, our framework may be viewed as an interpolation of special cases of Connes-Consani's $\mathbb{F}_1$-schemes[1] by special cases of Lorscheid's blue schemes[2].

Definitions. A **partial monoid** is a set A, a subset A_2 of $A \times A$ and a map $+\colon A_2 \to A$ such that $+$ is unital, commutative, and associative. For example, any based set is considered as a partial monoid by giving it a trivial structure — only the base point 0 can be added to other elements. For another example, a commutative monoid (thus a commutative group) is a partial monoid where A_2 is taken to be the whole set $A \times A$. A **partial ring** is a partial monoid with a bilinear, associative and commutative product $\cdot\colon A \times A \to A$ and identity $1 \in A$. For example, any commutative monoid with absorbing element 0 is a partial ring in which only 0 can be added to other elements, and any commutative semiring (thus a commutative ring) with identity is a partial ring. If A is a partial ring, an A-**module**, an **ideal** and a **prime ideal** is defined as in the usual commutative algebra, respectively. The set of prime ideals of a partial ring A is given the Zariski topology to make a topological space X and we can define a sheaf $\mathcal{O}_X$ of partial rings on X. Then $\operatorname{Spec} A = (X, \mathcal{O}_X)$ is the **affine partial scheme** of A. Then a **partial scheme** is defined to be a locally partial-ringed space that is locally an affine partial scheme.

Main result. Let $\mathcal{PRing}, \mathcal{PGrp}$, and $\mathcal{Grp}$ denote the category of partial rings, partial groups, and groups, respectively. We will give a definition of a good partial ring in the talk. Commutative monoids with absorbing element 0 and commutative rings with identity are examples of good partial rings.

Theorem 1. *There exists a representable functor $\mathbb{GL}_n\colon \mathcal{PRing} \to \mathcal{PGrp}$ which enjoys the following properties:*

(1) *its restriction to the category of good partial rings factors through $\mathcal{Grp}$.*
(2) $\mathbb{GL}_n(A)$ *is the group of n-th general linear group with entries in A, if A is a commutative rings with 1, and*
(3) $\mathbb{GL}_n(\mathbb{F}_1) = \mathfrak{S}_n$ *is n-th symmetric group.*

The reader will find details of this talk in the preprint [arXiv:2206.06084].

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COVERING THEORY OF SILTING OBJECTS

Takahiro Honma

The covering technique has been playing an important role in representation theory of finite dimensional algebras over an algebraically closed field K. By using this technique, we can reduce problems of the module category of an algebra Λ to the more easier category called an orbit category. The orbit category $\mathcal{C}/G$ is constructed by a category $\mathcal{C}$ with an action of a group G. In the original setting, covering theory was not possible to apply to general linear categories. This restriction was removed in [2, 3], and covering theory can be applied for several categories such as derived categories, singular categories, and Gorenstein defect categories.

In this talk, we apply the covering technique to silting theory. Silting theory is a generalization of tilting theory that deals with the equivalence of triangulated categories. As a previous study, tilting theory using the covering technique has been studied by Asashiba, who showed that the covering functor preserves the tilting subcategory. Our aim is to extend this claim to silting and also we consider the structure of silting subcategories(objects).

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The Grothendieck monoid of an extriangulated category

Haruhisa Enomoto

This talk is based on joint work with Shunya Saito (Nagoya University). The *Grothendieck group* is the classical and basic invariant for both a triangulated category and an exact category. For exact categories, the *Grothendieck monoid*, a natural monoid version of the Grothendieck group, has been recently studied by several authors [1, 2, 6].

In the representation theory of algebras, we often consider extension-closed subcategories of a triangulated category which are not exact nor triangulated. An *extriangulated category* introduced by Nakaoka–Palu [5] is a convenient framework to consider such subcategories. Extriangulated categories unify both exact categories and triangulated categories, and have the notion of *conflations*, which generalize conflations (short exact sequences) in an exact category and triangles in a triangulated category. We can naturally define the *Grothendieck monoid* $\mathsf{M}(\mathcal{C})$ of an extriangulated category $\mathcal{C}$ using conflations. In this talk, we give several results about it.

The first result is about the classifications of several classes of subcategories, which extends [6] and [7] respectively.

Theorem 1. *Let $\mathcal{C}$ be an extriangulated category. Then we have the following two bijections.*

(1) *A bijection between the set of Serre subcategories of $\mathcal{C}$ and the set of faces of $\mathsf{M}(\mathcal{C})$.*
(2) *A bijection between the set of dense 2-out-of-3 subcategories and the set of cofinal subtractive submonoids of $\mathsf{M}(\mathcal{C})$.*

The second result is about the localization of an extriangulated category. For a nice subcategory $\mathcal{N}$ of an extriangulated category $\mathcal{C}$, Nakaoka–Ogawa–Sakai [4] constructed the *exact localization* $\mathcal{C}/\mathcal{N}$, which generalizes the Verdier quotient of a triangulated category and the Serre quotient of an abelian category. We show that under some conditions, this *commutes with the Grothendieck monoid*:

Theorem 2. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a subcategory of $\mathcal{C}$ satisfying some conditions. Then we have an isomorphism of monoids*

$$\mathsf{M}(\mathcal{C}/\mathcal{N}) \cong \mathsf{M}(\mathcal{C})/\mathsf{M}_{\mathcal{N}},$$

where the right hand side is the monoid quotient by $\mathsf{M}_{\mathcal{N}} := \{[N] \mid N \in \mathcal{N}\}$. This can be applied to the Verdier quotient of a triangulated category, the stable category of a Frobenius category, and the Serre quotient of an abelian category.

As a toy example, we consider an *intermediate subcategory* of the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$, which is a subcategory $\mathcal{C}$ closed under extensions and direct summand satisfying $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A}$. We show that an intermediate subcategory is precisely a subcategory of the form $\mathcal{F}[1] * \mathcal{A}$ for a torsionfree class $\mathcal{F}$ of $\mathcal{A}$, and then compute its Grothendieck group, classify Serre subcategories, and study the exact localization.

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A bijection between silting subcategories and bounded hereditary cotorsion pairs

Takahide Adachi, Mayu Tsukamoto

Silting subcategories were introduced in ([5]) to study bounded t-structures and later studied in ([2]) from the viewpoint of mutation theory. It is known that silting subcategories are closely related to algebraic t-structures and simple-minded collections for finite dimensional algebras ([6]). On the other hand, Bondarko ([4]) and Pauksztello ([9]) independently introduced co-t-structures as an analog of t-structures. Bondarko ([4]) and Mendoza–Santiago–Sáenz–Souto ([7]) gave the following result.

Theorem 1 ([4, 7])**.** *Let $\mathcal{D}$ be a triangulated category. Then there exist mutually inverse bijections between the set of silting subcategories of $\mathcal{D}$ and the set of bounded co-t-structures of $\mathcal{D}$.*

The aim of this talk is to generalize Theorem 1 to extriangulated categories introduced by Nakaoka–Palu ([8]) as a simultaneous generalization of a triangulated category and an exact category. To give a generalization of Theorem 1, we introduce the notion of silting subcategories in an extriangulated category. The following theorem is a main result of this talk.

Theorem 2 ([1])**.** *Let $\mathcal{C}$ be an extriangulated category. Then there exist mutually inverse bijections between the set of silting subcategories of $\mathcal{C}$ and the set of bounded hereditary cotorsion pairs in $\mathcal{C}$.*

For a triangulated category $\mathcal{D}$, we can naturally regard $\mathcal{D}$ as an extriangulated category and bounded co-t-structures of a triangulated category $\mathcal{D}$ coincide with bounded hereditary cotorsion pairs in an extriangulated category $\mathcal{D}$. Hence Theorem 1 follows from Theorem 2.

Let A be an artin algebra with finite global dimension. Then we can naturally regard $\mathsf{mod}\,A$ as an extriangulated category and tilting A-modules coincide with silting objects of an extriangulated category $\mathsf{mod}\,A$. Thus Theorem 2 also recovers the following result.

Corollary 3 ([3])**.** *Let A be an artin algebra with finite global dimension. Then $T \mapsto {}^{\perp}T$ gives a bijection between the set of isomorphism classes of basic tilting A-modules and the set of contravariantly finite resolving subcategories of $\mathsf{mod}\,A$, and $T \mapsto T^{\perp}$ gives a bijection between the set of isomorphism classes of basic tilting A-modules and the set of covariantly finite coresolving subcategories of $\mathsf{mod}\,A$.*

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Dimitrov–Haiden–Katzarkov–Kontsevich complexities for singularity categories

Ryo Takahashi

In 2014, Dimitrov, Haiden, Katzarkov and Kontsevich [2] introduced the notions of complexities and entropies for a triangulated category. In less than a decade since then, a lot of works on these notions have been done; see [3, 4, 5, 6, 7, 8, 9, 12] for instance. Let us recall the definitions.

Definition 1 (Dimitrov–Haiden–Katzarkov–Kontsevich)**.** Let $\mathcal{T}$ be a triangulated category.

(1) Let $A, B \in \mathcal{T}$ and $t \in \mathbb{R}$. We denote by $\delta_t(A, B)$ the infimum of the sums $\sum_{i=1}^{r} e^{n_i t}$, where r runs through the nonnegative integers, and $n_1, \dots, n_r$ run through the integers such that there exists a series

$$\{ B_{i-1} \to B_i \to A[n_i] \rightsquigarrow \}_{i=1}^{r}$$

of exact triangles in $\mathcal{T}$ with $B_0 = 0$ and B_r containing B as a direct summand. The function

$$\mathbb{R} \ni t \mapsto \delta_t(A, B) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

is called the *(Dimitrov–Haiden–Katzarkov–Kontsevich) complexity* of B relative to A.

(2) Let $F : \mathcal{T} \to \mathcal{T}$ be an exact functor and $t \in \mathbb{R}$. The *entropy* $\mathrm{h}_t(F)$ of F is defined by

$$\mathrm{h}_t(F) = \lim_{n\to\infty} \frac{1}{n} \log \delta_t(G, F^n(G)),$$

where G is a *split generator* of $\mathcal{T}$, i.e., G is an object of $\mathcal{T}$ whose thick closure coincides with $\mathcal{T}$.

Let R be a commutative noetherian local ring. Let $\mathsf{D}_{\mathsf{sg}}(R)$ be the *singularity category* of R, which is a triangulated category introduced by Buchweitz [1] and Orlov [10]. In this talk, we explore complexities for $\mathsf{D}_{\mathsf{sg}}(R)$. More specifically, we shall consider the following question.

Question 2. Let G be a split generator of $\mathsf{D}_{\mathsf{sg}}(R)$. Then does it hold that

$$\delta_t(G, X) = 0$$

for all $X \in \mathsf{D}_{\mathsf{sg}}(R)$ and $t \neq 0$?

The contents of this talk will basically come from [11].

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Categorical entropy of the Frobenius pushforward functor

Hiroki Matsui

This talk is based on the ongoing joint work with Ryo Takahashi.

For a *categorical dynamical system*, i.e., a pair $(\mathcal{T}, \Phi)$ of a triangulated category $\mathcal{T}$ and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov, Haiden, Katzarkov and Kontsevich [1] introduced an invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the *categorical entropy* of Φ as a categorical analogue of the topological entropy. The categorical entropy $h_t^{\mathcal{T}}(\Phi)$ is a function in one real variable t with values in $\mathbb{R} \cup \{-\infty\}$ and measures the complexity of the exact endofunctor Φ.

For a commutative noetherian local ring with positive characteristic p, the ring endomorphism $F : R \to R$, which is called the *Frobenius endomorphism*, is defined by $F(a) = a^p$. Assume further that $F : R \to R$ is module finite. The Frobenius endomorphism F induces two exact endofunctors: the *Frobenius pushforward*

$$\mathbb{R}F_* = F_* : \mathrm{D^b}(R) \to \mathrm{D^b}(R)$$

on the bounded derived category $\mathrm{D^b}(R)$ of finitely generated R-modules and the *Frobenius pullback*

$$\mathbb{L}F^* : \mathrm{D^{perf}}(R) \to \mathrm{D^{perf}}(R)$$

on the derived category $\mathrm{D^{perf}}(R)$ of perfect R-complexes. Both these functors are main tools to study singularities in positive characteristics.

For the Frobenius pullback, Majidi-Zolbanin and Miasnikov [2] considered the full subcategory $\mathrm{D^{perf}_{\mathfrak{m}}}(R)$ of $\mathrm{D^{perf}}(R)$ consisting of perfect complexes with finite length cohomologies, and computed the categorical entropy $h_t^{\mathrm{D^{perf}_{\mathfrak{m}}}(R)}(\mathbb{L}F^*)$:

Theorem 1. [2, Corollary 2.6] *Let R be a d-dimensional commutative complete noetherian local ring with positive characteristic p. Then the equality*

$$h_t^{\mathrm{D^{perf}_{\mathfrak{m}}}(R)}(\mathbb{L}F^*) = d \log p$$

holds.

In this talk, we study the Frobenius pushforward F_* and consider its categorical entropy $h_t^{\mathrm{D^b}(R)}(F_*)$. The main result of this talk is the following:

Theorem 2. *Let R be a d-dimensional commutative noetherian local ring with positive characteristic p. Then the equality*

$$h_t^{\mathrm{D^b}(R)}(F_*) = d \log p + \log[F_*(k) : k]$$

holds, where k denotes the residue field of R.

We will also discuss the relation between the categorical entropy $h_t^{\mathrm{D^b}(R)}(\phi_*)$ of the pushforward functor along a local ring endomorphism $\phi : R \to R$ and the *local entropy* $h_{loc}(\phi)$ of ϕ which has been introduced in [3].

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The reduction number of stretched ideals

Kazuho Ozeki

The homological property of the associated graded ring of an ideal is an important problem in commutative algebra. In this talk we explore the almost Cohen-Macaulayness of the associated graded ring of stretched $\mathfrak{m}$-primary ideals.

Throughout this talk, let A be a Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. For simplicity, we may assume the residue class field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let

$$R = \mathrm{R}(I) := A[It] \quad \subseteq A[t] \text{ and } R' = \mathrm{R}'(I) := A[It, t^{-1}] \quad \subseteq A[t, t^{-1}]$$

denote, respectively, the Rees algebra and the extended Rees algebra of I. Let

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}$$

denote the associated graded ring of I. Let $Q = (a_1, a_2, \cdots, a_d) \subseteq I$ be a parameter ideal in A which forms a reduction of I. We set

$$n_I = n_Q(I) := \min\{n \geq 0 \mid I^{n+1} \subseteq Q\} \text{ and } r_I = r_Q(I) := \min\{n \geq 0 \mid I^{n+1} = QI^n\},$$

respectively, denote the index of nilpotency and the reduction number of I with respect to Q. Then it is easy to see that the inequality $r_I \geq n_I$ always holds true. Let $\ell_A(N)$ denote, for an A-module N, the length of N.

The notion of *stretched* Cohen-Macaulay local rings was introduced by J. Sally [2]. Thereafter, in 2001, Rossi and Valla [1] gave the notion of stretched $\mathfrak{m}$-primary ideals. We say that the $\mathfrak{m}$-primary ideal I is stretched if the following two conditions

(1) $Q \cap I^2 = QI$ and
(2) $\ell_A(I^2 + Q/I^3 + Q) = 1$

hold true for some parameter ideal Q in A which forms a reduction of I (c.f. [1]). We notice that the first condition is naturally satisfied if $I = \mathfrak{m}$ so that this extends the classical definition of stretched local rings given in [2].

Sally [2] (for the case where $I = \mathfrak{m}$), and Rossi-Valla [1] showed that the equality $r_I = n_I$ holds true if and only if the associated graded ring G is Cohen-Macaulay in the case where I is stretched. Thus stretched $\mathfrak{m}$-primary ideals whose reduction number attends to minimal value enjoy nice properties.

The purpose of this talk is to explore the almost Cohen-Macaulayness of associated graded ring of stretched $\mathfrak{m}$-primary ideal I in the case where the reduction number attains almost minimal value. As an application, we give almost Cohen-Macaulayness of the associated graded ring of stretched $\mathfrak{m}$-primary ideals with reduction number at most four as follows.

Corollary 1. *Suppose that I is stretched and assume that $I^5 = QI^4$ (i.e. $r_I \leq 4$), then G is almost Cohen-Macaulay (i.e. $\operatorname{depth} G \geq d - 1$).*

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Higher versions of morphisms represented by monomorphisms

Yuya Otake

Throughout this talk, let R be a two-sided noetherian ring. We assume that all modules are finitely generated right ones. It is a natural and classical question to ask when a given homomorphism of R-modules is stably equivalent to another homomorphism satisfying certain good properties. A homomorphism $f : X \to Y$ of R-modules is said to be *represented by monomorphisms* if there is an R-homomorphism $t : X \to P$ with P projective such that $\binom{f}{t} : X \to Y \oplus P$ is a monomorphism. This notion has been introduced by Auslander and Bridger [1], and played an important role in the stable module theory they developed. Recently, Kato [6] gave various characterizations of the morphisms represented by monomorphisms.

The notion of n-torsionfree modules was also introduced by Auslander and Bridger [1], and played a central role in the theory they developed. For example, for an R-module M, Auslander and Bridger figured out the relationship between the grade of the Ext module $\mathrm{Ext}^i_R(M, R)$ and the torsionfreeness of the syzygy $\Omega^i M$. The structure of n-torsionfree modules has been well-studied; see [1, 2, 3, 4, 5, 7]. In this talk, we introduce and study a new class of morphisms which includes morphisms represented by monomorphisms. As an application, we give a common generalization of several results due to Auslander and Bridger that describe relationships between torsionfreeness and the grades of Ext modules.

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On the openness of loci over Noetherian rings

Kaito Kimura

Let R be a commutative noetherian ring, and $\mathbb{P}$ a property of commutative local rings. There is a topology on $\mathrm{Spec}(R)$, which is called the *Zariski topology.* The set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_{\mathfrak{p}}$ satisfies $\mathbb{P}$ is called the $\mathbb{P}$-locus of R. It is a natural question to ask when the $\mathbb{P}$-locus is open in the Zariski topology for a given $\mathbb{P}$. This question has been studied for a long time by many people.

Nagata [6] produced the following condition, which is called the *Nagata criterion*: if the $\mathbb{P}$-locus of $R/\mathfrak{p}$ contains a nonempty open subset of $\mathrm{Spec}(R/\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R, then the $\mathbb{P}$-locus of R is an open subset of $\mathrm{Spec}(R)$. It is well-known fact that the Nagata criterion holds for the regular, complete intersection, Gorenstein, and Cohen–Macaulay properties and Serre's conditions; see [1, 5, 6].

Let M be a finitely generated R-module, and $\mathbb{Q}$ a property of modules over a commutative local ring. The set of prime ideals $\mathfrak{p}$ of R such that the module $M_{\mathfrak{p}}$ over the local ring $R_{\mathfrak{p}}$ satisfies $\mathbb{Q}$ is called the $\mathbb{Q}$-locus of M (over R). The same question can be asked for the $\mathbb{Q}$-locus of M. The Cohen–Macaulay locus of a module over an excellent ring is open [2]. Furthermore, the Gorenstein locus of a module over an acceptable ring is open [4], and so is the finite injective dimension locus of a module over an excellent ring [7].

In this talk, we consider the openness of the $\mathbb{Q}$-locus of a module in the case where $\mathbb{Q}$ is each of the finite injective dimension property, the Gorenstein property, the Cohen–Macaulay property, the maximal Cohen–Macaulay property, and Serre's condition. We give a module version of the Nagata criterion for properties of modules, and show that it holds for the properties appearing above.

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A characterization of standard derived equivalences of diagrams of dg categories and their gluing

Hideto Asashiba and Shengyong Pan

Throughout this talk $\Bbbk$ is a commutative ring, and I is a small category. A *dg category* (a short form of *differential graded category*) is a $\Bbbk$-category $\mathcal{A}$ whose morphism spaces $\mathcal{A}(x, y)$ are (cochain) complexes of $\Bbbk$-modules, and whose compositions $\mathcal{A}(y, z) \times \mathcal{A}(x, y) \to \mathcal{A}(x, z)$ are chain maps of complexes. We denote by $\Bbbk$-**dgCAT** the 2-category of *light* (namely usual) dg categories, dg functors between them, and dg natural transformations between those dg functors. The 2-category $\Bbbk$-**DGCAT** has the same objects and 1-morphisms as $\Bbbk$-**dgCAT**, but its 2-morphisms are the derived transformations between its 1-morphisms. Finally, $\Bbbk$-**dgCat** (resp. $\Bbbk$-**DGCat**) denotes the full 2-subcategory of $\Bbbk$-**dgCAT** (resp. $\Bbbk$-**DGCAT**) consisting of the small dg categories. We formulate a diagram of dg categories and dg functors as a colax functor X from I to $\Bbbk$-**dgCat**. We can also regard X as a set of dg categories $X(i)$'s with an action of I, hence as a generalization of a dg category with a group action when I is a group viewed as a category with only one object $*$. For a 2-category $\mathbf{C}$, the colax functors from I to $\mathbf{C}$ also form a 2-category $\mathrm{Colax}(I, \mathbf{C})$ with suitably defined 1-morphisms and 2-morphisms.

For a colax functor X in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, a dg category $\int X$ can be constructed by "gluing" all dg categories $X(i)$'s together, which is called the *Grothendieck construction* of X, which is nothing but the orbit category $X(*)/G$ when I is a group G.

Let $\mathcal{C}_{dg}(\Bbbk)$ be the dg category of the complexes over $\Bbbk$. Then a contravariant dg functor from a small dg category $\mathcal{A}$ to $\mathcal{C}_{dg}(\Bbbk)$ is called a (right) *dg $\mathcal{A}$-module*. They form a (light) *dg category* $\mathcal{C}_{dg}(\mathcal{A})$ whose morphisms are the derived transformations between these objects as dg functors. By taking the 0-cocycle, we obtain the *category* $\mathcal{C}(\mathcal{A}) := Z^0(\mathcal{C}_{dg}(\mathcal{A}))$ of dg $\mathcal{A}$-modules, which is known to be a (light) Frobenius category. Similarly by taking the 0-homology, we obtain the *homotopy category* $\mathcal{H}(\mathcal{A}) := H^0(\mathcal{C}_{dg}(\mathcal{A}))$ of $\mathcal{A}$, or alternatively by taking the stable category we obtain the same one $\mathcal{H}(\mathcal{A}) := \underline{\mathcal{C}(\mathcal{A})}$, which is known to be a (light) triangulated category. Finally, the derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is defined as the quotient of $\mathcal{H}(\mathcal{A})$ by formally inverting the quasi-isomorphisms, which is known to be a (2-moderate) triangulated category. The correspondence $\mathcal{A} \mapsto \mathcal{C}_{dg}(\mathcal{A})$ can be extended to a pseudofunctor $\mathcal{C}_{dg}\colon \Bbbk\text{-}\mathbf{DGCat} \to \Bbbk\text{-}\mathbf{DGCAT}$. Similarly, we obtain pseudofunctors $\mathcal{C}\colon \Bbbk\text{-}\mathbf{dgCAT} \to \Bbbk\text{-}\mathbf{FRB}$, $\mathcal{H}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$, and $\mathcal{D}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\mathbf{TRI}}^2$. For a colax functor $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$, we can define its dg category of dg modules $\mathcal{C}_{dg}(X)$, category of dg modules $\mathcal{C}(X)$, homotopy category $\mathcal{H}(X)$, and derived category $\mathcal{D}(X)$ as the composite $\mathcal{C}_{dg}(X) := \mathcal{C}_{dg} \circ X$ and so on.

We obtained the following characterization of "standard derived equivalences" between diagrams of dg categories. Note that we do not need $\Bbbk$-flatness assumption unlike a result by Keller [2].

Theorem 1. *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then the following are equivalent:*

(1) *There exists a 1-morphism $(F, \psi)\colon \mathcal{C}_{dg}(X') \to \mathcal{C}_{dg}(X)$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that the left derived functor $\mathbf{L}(F, \psi)\colon\ \mathcal{D}(X') \to \mathcal{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.*

(2) *There exists a quasi-equivalence $(E, \phi)\colon X' \to \mathcal{T}$ for some tilting colax functor $\mathcal{T}$ for X.*

The following gives a sufficient condition for the Grothendieck constructions to be derived equivalent.

Theorem 2. *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. If there exists a quasi-equivalence $(E, \phi)\colon X' \to \mathcal{T}$ for some tilting colax functor $\mathcal{T}$ for X, then $\int X$ and $\int X'$ are derived equivalent.*

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Localization of triangulated categories with respect to extension-closed subcategories

Yasuaki Ogawa

In this talk, we develop a framework for localization theory of triangulated categories $\mathcal{C}$ with respect to an extension-closed subcategory $\mathcal{N}$ which is not necessarily thick [4]. The obtained "exact" functor $Q : \mathcal{C} \to \widetilde{\mathcal{C}}_{\mathcal{N}}$ to the quotient category provides a unification of the Verdier quotient and some cohomological functors, e.g., the heart of t-structures, the abelian quotient by 2-cluster tilting subcategories and a more general phenomenon, the heart of cotorsion pairs. To deal with the triangulated structure on $\mathcal{C}$ and (abelian) exact structures on $\widetilde{\mathcal{C}}_{\mathcal{N}}$ in the same manner, we formulate the results through Nakaoka-Palu's extriangulated structures [2]. Our construction of the exact functor $Q : \mathcal{C} \to \widetilde{\mathcal{C}}_{\mathcal{N}}$ is divided in two steps:

- *Relative theory* [1]: There exists a weaken structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ relative to the triangulated category $\mathcal{C}$ which is determined by $\mathcal{N}$.
- *Localization* [3]: Then, $\mathcal{N}$ becomes a "thick" subcategory in $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$, which enables us to define the quotient category of $\mathcal{C}$ by $\mathcal{N}$.

Thus, in general, our resulting category $\widetilde{\mathcal{C}}_{\mathcal{N}}$ is an extriangulated category. Furthermore, we provide necessary and sufficient conditions for $\mathcal{N}$ to make $\widetilde{\mathcal{C}}_{\mathcal{N}}$ to be triangulated and abelian. Our results are summarized as follows.

Theorem 1. *Let $\mathcal{C}$ be a triangulated category and regard it as a natural extriangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Assume that a full subcategory $\mathcal{N}$ of $\mathcal{C}$ is closed under direct summands, isomorphisms and extensions.*

(1) *The subcategory $\mathcal{N}$ is thick with respect to the relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$. Moreover, we have an extriangulated localization $(Q, \mu) : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\widetilde{\mathcal{C}}_{\mathcal{N}}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$.*

(2) *The subcategory $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if the resulting category $\widetilde{\mathcal{C}}_{\mathcal{N}}$ is triangulated. In this case, the localization (Q, μ) is nothing but the Verdier quotient.*

(3) *Suppose that $\mathcal{N}$ is functorially finite. Then, $\mathcal{N}$ satisfies $\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$ in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if the resulting category $\widetilde{\mathcal{C}}_{\mathcal{N}}$ is abelian. Furthermore, the functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \widetilde{\mathcal{C}}_{\mathcal{N}}$ from the original triangulated category is cohomological.*

$\mathcal{N}$	*extension-closed*	*thick*	$\mathsf{Cone}(\mathcal{N}, \mathcal{N}) = \mathcal{C}$
$\widetilde{\mathcal{C}}_{\mathcal{N}}$	*extraingulated*	*triangulated*	*abelian*

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Projective objects in the category of discrete modules over a profinite group

Ryo Kanda

This talk is based on a joint work with Alexandru Chirvasitu [CK19].

It is known that $\operatorname{Mod} R$ and $\operatorname{QCoh} X$ are both Grothendieck categories, where $\operatorname{Mod} R$ is the category of (left) modules over a ring R and $\operatorname{QCoh} X$ is the category of quasi-coherent sheaves on a scheme X. In particular, they both have exact direct limits (and hence exact direct sums) and enough injectives. $\operatorname{Mod} R$ also has exact direct products and enough projectives, while it is known that none of these holds for $\operatorname{QCoh} X$ when X is a non-affine divisorial noetherian scheme ([Kan19]).

We consider a similar question concerning the category of discrete modules over a profinite group. Our main result is the following:

Theorem 1 ([CK19])**.** *Let G be a profinite group. Then the following conditions are equivalent:*

(1) *The category of discrete G-modules has enough projectives.*
(2) *The category of discrete G-modules has exact direct products.*
(3) *G is a finite group.*

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