Characterizations of standard derived equivalences of diagrams of dg categories and their gluing

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1. Motivation

$\Bbbk$: alg. closed field

Representation-finite selfinjective algebras

standard

non-standard

only char $\Bbbk = 2$

der. eq. type : $(D_{3m}, 1/3, 1)$

$(m \geq 2)$

$$A \cong \hat{B}/G$$

B : a tilted alg of Dynkin

$\hat{B}$:= the repetitive cat of B,

having G-action (G : infin cyclic gp)

(∵) $\exists$ a G-covering $\hat{B} \xrightarrow{P} A$

A' : another such alg with the G-covering $\hat{B}' \xrightarrow{P'} A'$

Main tools

$$\begin{cases} (1) & B \overset{\text{der}}{\sim} B' \Longrightarrow \hat{B} \overset{\text{der}}{\sim} \hat{B}' \\ (2) & \hat{B} \overset{\text{der}}{\sim} \hat{B}' + \text{some compatibility with } P, P' \Longrightarrow A \overset{\text{der}}{\sim} A' \end{cases}$$

(2) is generalized (2013): $\Bbbk$: any commutative ring

G: a cyclic gp

$\mathcal{C} = \hat{B}$: locally bdd $\Bbbk$-cat

G-action $X: G \to \mathrm{Aut}(\mathcal{C})$

$\wr$

$X: G \to \Bbbk\text{-Cat}$

$* \mapsto \mathcal{C}$

$\mathcal{C}/G$

$\mathrm{Mod}\,\mathcal{C}$

$\mathcal{D}(\mathrm{Mod}\,\mathcal{C})$

$\mathcal{C} \overset{\text{der}}{\sim} \mathcal{C}' :\Longleftrightarrow \mathcal{D}(\mathrm{Mod}\,\mathcal{C}) \simeq \mathcal{D}(\mathrm{Mod}\,\mathcal{C}')$ as tri cats

I: a small cat

$\mathcal{C}$: a small or light $\Bbbk$-cat

(colax) functor $X: I \to \Bbbk\text{-Cat}$ — the 2-cat of small $\Bbbk$-cats

eg. $I = \text{free}(1 \xrightarrow{a} 2 \xrightarrow{b} 3)$

$X(a): X(1) \xrightarrow{X(a)} X(2) \xrightarrow{X(b)} X(3)$ in $\Bbbk\text{-Cat}$

$\int X$:= the Grothendieck construction of X

$\mathrm{Mod}\,X$

$\mathcal{D}(\mathrm{Mod}\,X): \underbrace{I \xrightarrow{X} \Bbbk\text{-Cat} \xrightarrow{\mathrm{Mod}} \Bbbk\text{-ModCat}}_{\mathrm{Mod}\,X}$

$\downarrow \mathcal{D}$

$\Bbbk\text{-}\underline{\mathrm{TRI}}^2$

$X \overset{\text{der}}{\sim} X' :\Longleftrightarrow \mathcal{D}(\mathrm{Mod}\,X) \simeq \mathcal{D}(\mathrm{Mod}\,X')$ as colax funs

Q1. Define and characterize derived eq. for X, X'

Q2. When $\int X' \overset{\mathrm{der}}{\simeq} \int X$?

A1 $X' \overset{\mathrm{der}}{\simeq} X \;:\Longleftrightarrow\; \exists (F, \phi) : \mathcal{D}(\mathrm{Mod}\, X') \longrightarrow \mathcal{D}(\mathrm{Mod}\, X)$

an eq in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{Cat})$

$$\overset{\mathrm{Prp}}{\Longleftrightarrow} \begin{cases} \forall i \in I_0,\ F(i) : \text{triangle eq} \\ \forall a \in I_1,\ \phi_a : \text{nat iso} \end{cases}$$

$\boxed{\begin{array}{l} \textbf{Thm 1} \\ (1)\quad X' \overset{\mathrm{der}}{\simeq} X \\ \quad\Downarrow \\ (2)\quad X' \xrightarrow[\mathrm{eq}]{(F', \phi')} \mathcal{T} \xrightarrow{(\sigma, \rho)} K^b(\mathrm{prj}\, X) \\ X : \Bbbk\text{-flat} \qquad \forall i \in I_0,\ \mathcal{T}(i) : \text{tilting subcat of } K^b(\mathrm{prj}\, X(i)). \end{array}}$

X : <u>$\Bbbk$-flat</u> $:\Longleftrightarrow$ $X(i)(x, y) : \Bbbk\text{-flat}$ $(i \in I_0,\ x, y \in X(i)_0)$

(2)

$$\begin{tikzcd}
X'(i) \arrow[r, "F(i)", "\sim"'] \arrow[d, "X'(a)"'] \arrow[dr, phantom, "\cong\swarrow \phi_a"] & T(i) \arrow[r, hook, "\sigma(i)"] \arrow[d] \arrow[dr, phantom, "\cong\swarrow \rho_a"] & K^b(\mathrm{proj}\, X(i)) \arrow[d, "K^b(\mathrm{proj}\, X(a))"] \\
X'(j) \arrow[r, "\sim", "F(j)"'] & T(j) \arrow[r, hook, "\sigma(j)"'] & K^b(\mathrm{proj}\, X(j))
\end{tikzcd}$$

tilting

A2

$$\boxed{\underline{\text{Thm 2}} \quad (2) \Rightarrow \int X' \overset{\text{der}}{\sim} \int X.}$$

<u>Problem</u> Generalize these to dg cat.s

Characterization of der eq is well controlled in the setting of dg cat.s

[Keller: Deriving DG categories]

2. Collection of categories

- conglomerate
- classes
- sets

$\mathcal{U} \subset \mathcal{U}' \subseteq \cdots$

(Grothendieck) universes

- sets
- ⋮
- $\mathcal{U}'$-classes
- $\mathcal{U}'$-small sets
- $\mathcal{U}$-classes
- $\mathcal{U}$-small sets

← refine

$\forall S$: a set, S: $\underline{\mathcal{U}\text{-small}}$ $:\Leftrightarrow S \in \mathcal{U}$

S: $\underline{\mathcal{U}\text{-class}}$ $:\Leftrightarrow S \subseteq \mathcal{U}$

Fix only one universe $\mathcal{U} (\ni \mathbb{N})$

- $\mathcal{U}'$-small sets
- ⋮
- 3-classes
- 2-classes
- 1-classes $= (\mathcal{U}\text{-})$classes
- 0-classes $= (\mathcal{U}\text{-})$small sets

Enough for today's talk

remove "$\mathcal{U}$" from names

Dfn. $\mathcal{C}$: a cat

$\mathcal{C}$: $\underline{\text{small}}$ $:\Leftrightarrow \begin{cases} \mathcal{C}_0 : \text{small} \\ \mathcal{C}(x,y) : \text{small} \quad \forall x,y \in \mathcal{C}_0 \end{cases}$ — **Cat**

$\mathcal{C}$: $\underline{\text{light}}$ $:\Leftrightarrow \begin{cases} \mathcal{C}_0 : \text{1-class} \\ \mathcal{C}(x,y) : \text{small} \quad \forall x,y \in \mathcal{C}_0 \end{cases}$ — **CAT**

$\mathcal{C}$: $\underline{\text{1-moderate}}$ $:\Leftrightarrow \begin{cases} \mathcal{C}_0 : \text{1-class} \\ \mathcal{C}(x,y) : \text{1-class} \quad \forall x,y \in \mathcal{C}_0 \end{cases}$ — $\underline{\mathbf{CAT}}$

$k \geq 1$

$\mathcal{C}$: $\underline{k\text{-moderate}}$ $:\Leftrightarrow \left.\begin{matrix} \mathcal{C}_0 \\ \mathcal{C}(x,y) \end{matrix}\right\} k\text{-class}$ — $\underline{\mathbf{CAT}}^k$

2-cats

2. Preparation

1-morphism

obj

2-morphism

$$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd}$$

<u>Dfn</u> A <u>2-category</u> is a sequence of data

(1) $\mathbb{C}_0 \neq \emptyset$: a set

(2) $(\mathbb{C}(x,y))_{x,y\in\mathbb{C}_0}$: a family of categories

(3) $\circ = (\circ_{x,y,z} : \mathbb{C}(y,z)\times\mathbb{C}(x,y) \longrightarrow \mathbb{C}(x,z))_{x,y,z\in\mathbb{C}_0}$

a family of functors

(4) $\mathbb{1} = (u_x : \mathbb{1} \to \mathbb{C}(x,x))_{x\in\mathbb{C}_0}$: a family of functors

$* \circlearrowright 1_*$ $\qquad$ $1_x := u_x(*)$, $1_{1_x} := u_x(1_*)$

that satisfies associativity and unitality.

<u>Exm</u> (1) $\mathbb{k}\text{-}\mathbb{C}\mathrm{at}$: $\begin{cases} \cdot \text{ small } \mathbb{k}\text{-cats} \\ \cdot\ \mathbb{k}\text{-functors} \\ \cdot \text{ natural transformations} \end{cases}$

(2) $\mathcal{C}$: a cat. $\mathcal{C}$ is regarded as a 2-cat $\begin{cases} \cdot \text{ objects of } \mathcal{C} \\ \cdot \text{ morphisms of } \mathcal{C} \\ \cdot\ 1_f \ \left(\begin{matrix} f\in\mathcal{C}(x,y) \\ x,y\in\mathcal{C}_0 \end{matrix} \right) \end{cases}$

Dfn $\mathbb{A}, \mathbb{B}$: 2-cat^s. A <u>2-functor</u> $X: \mathbb{A} \to \mathbb{B}$ is a pair of data

(1) $X_0 : \mathbb{A}_0 \to \mathbb{B}_0$: a map $\quad X(x) := X_0(x) \ (x \in \mathbb{A}_0)$

(2) $\left(X_{(x,y)} : \mathbb{A}(x,y) \to \mathbb{B}(x,y)\right)_{x,y \in \mathbb{A}_0}$: a family of functors $\quad X(f) := X_{(x,y)}(f) \ (f \in \mathbb{A}(x,y))$

$X(gf) = X(g)X(f)$

$X(1_x) = 1_{X(x)}$

that preserves compositions and identities.

Dfn $\mathbb{A}, \mathbb{B}$: 2-cat^s. A <u>colax functor</u> $X: \mathbb{A} \to \mathbb{B}$ is a sequene of data

(1), (2) as above

(3) $(X_x : X(1_x) \Rightarrow 1_{X(x)})_{x \in \mathbb{A}_0}$ a family of 2-mor^s

(4) $(X_{b,a} : X(ba) \Rightarrow X(b)X(a))_{\leftarrow b \leftarrow a \leftarrow \text{ in } \mathbb{A}}$ a fam of 2-mor^s, natural in a, b

that satisfies the axioms

(a) <u>counitality</u>

$$\begin{tikzcd} X(a1_x) \arrow[r, Rightarrow] \arrow[dr, equal] & X(a)X(1_x) \arrow[d, Rightarrow] \\ & X(a)1_{X(x)} \arrow[u, phantom, "\circlearrowleft", pos=0.9] \end{tikzcd} \qquad \begin{tikzcd} X(1_y a) \arrow[r, Rightarrow] \arrow[dr, equal] & X(1_y)X(a) \arrow[d, Rightarrow] \\ & 1_{X(y)}X(a) \arrow[u, phantom, "\circlearrowleft", pos=0.9] \end{tikzcd} \qquad \left(\begin{matrix} x \xrightarrow{a} y \\ \text{in } \mathbb{A} \end{matrix}\right)$$

(b) <u>coassociativity</u>

$$\begin{tikzcd} X(cba) \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\circlearrowleft"] & X(c)X(ba) \arrow[d, Rightarrow] \\ X(cb)X(a) \arrow[r, Rightarrow] & X(c)X(b)X(a) \end{tikzcd} \qquad \left(w \xleftarrow{c} z \xleftarrow{b} y \xleftarrow{a} x \text{ in } \mathbb{A}\right)$$

- A <u>lax functor</u> $X: \mathbb{A} \to \mathbb{B}$ is a colax fun $X: \mathbb{A} \to \mathbb{B}^{co}$
  ↑reverse the 2-mor of $\mathbb{B}$
- A <u>pseudofunctor</u> is a colax fun with
  all $X_x, X_{b,a}$: 2-iso's
- A 2-functor is nothing but a colax fun with all $X_i, X_{b,a}$ identities.

<u>Dfn</u> (1) $\mathcal{A}$: a cat

$\mathcal{A}$: a <u>dg cat</u> $:\Longleftrightarrow \begin{cases} \mathcal{A}(x,y): \text{a (cochain) complex of } \Bbbk\text{-modules } (x,y \in \mathcal{A}_0). \\ \mathcal{A}(y,z) \otimes_{\Bbbk} \mathcal{A}(x,y) \to \mathcal{A}(x,z): \text{a chain map } (x,y,z \in \mathcal{A}_0). \end{cases}$

(2) $\mathcal{A}, \mathcal{B}$: dg cat's. A <u>dg functor</u> $F: \mathcal{A} \to \mathcal{B}$ is a sequence of data

(i) $F_0: \mathcal{A}_0 \to \mathcal{B}_0$: a map $F(x) := F_0(x)$ $(x \in \mathcal{A}_0)$

(ii) $F_{(x,y)}: \mathcal{A}(x,y) \to \mathcal{B}(F(x), F(y))$: a chain map $(x, y \in \mathcal{A}_0)$

$F(f) := F_{(x,y)}(f)$ $(f \in \mathcal{A}(x,y))$

that preserves compositions and identities

<u>Dfn</u> (dg nat tr, derived tr) $\mathcal{A} \underset{F}{\overset{E}{\rightrightarrows}} \mathcal{B}$: dg cat^s and dg fun^s

$n \in \mathbb{Z}$.

- $\mathcal{H}om(E,F)^n := \left\{ (\alpha^n_x)_{x \in \mathcal{A}_0} \in \prod_{x \in \mathcal{A}_0} \mathcal{B}(Ex, Fx)^n \;\middle|\; \begin{tikzcd} Ex \arrow[r, "\alpha_x^n"] \arrow[d, "Ef"'] \arrow[dr, phantom, "(-1)^{mn}\swarrow"] & Fx \arrow[d, "Ff"] \\ Ey \arrow[r, "\alpha_y^n"'] & Fy \end{tikzcd} \quad \begin{array}{l} \forall f \in \mathcal{A}(x,y)^m \\ \forall m \in \mathbb{Z} \\ \forall x, y \in \mathcal{A}_0 \end{array} \right\}$
- $\mathcal{H}om(E,F) := \bigoplus_{n \in \mathbb{Z}} \mathcal{H}om(F,G)^n$
- $\mathcal{H}om(E,F)^n \longrightarrow \mathcal{H}om(E,F)^{n+1}$

$$(\alpha^n_x)_x \longmapsto (d_{\mathcal{B}}(\alpha^n_x))_x$$

$$\left\{ \begin{array}{l} \alpha^n = (\alpha^n_x)_{x \in \mathcal{A}_0} : \text{ a } \underline{\text{derived transformation of degree } n} \\ \alpha = (\alpha^n)_{n \in \mathbb{Z}} : \text{ a } \underline{\text{derived transformation}} \\ Z^0(\mathcal{H}om(E,F)) \ni \alpha \text{ is called a } \underline{\text{dg natural transformation}} \end{array} \right.$$

<u>i.e.</u> $\alpha_x \in \mathcal{B}(Ex, Fx)^0$, $d(\alpha) = 0$,

$$\begin{tikzcd} Ex \arrow[r, "\alpha_x"] \arrow[d, "Ef"'] \arrow[dr, phantom, "//"] & Fx \arrow[d, "Ff"] \\ Ey \arrow[r, "\alpha_y"'] & Fy \end{tikzcd}$$

α: a chain map

Dfn $\mathcal{C}_{dg}(\Bbbk) :=$ the cat of (co)chain complexes of $\Bbbk$-modules

$$\begin{cases} \mathcal{C}_{dg}(\Bbbk)(M,N) = \bigoplus_{n\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\Bbbk}(M^p, N^{p+n}) \\ d(f) := \left(d_N^{p+n} f^p - (-1)^n f^{p+1} d_M^p\right)_{p\in\mathbb{Z}} \quad \forall f = (f^p)_{p\in\mathbb{Z}} \in \mathcal{C}_{dg}(M,N)^n. \\ \begin{tikzcd} M^p \arrow[r, "d_M^p"] \arrow[d, "f^p"'] & M^{p+1} \arrow[d, "f^{p+1}"] \\ N^{p+n} \arrow[r, "d_N^{p+n}"'] & N^{p+n+1} \end{tikzcd} \end{cases}$$

$\mathcal{C}_{dg}(\Bbbk)$ is a light dg cat.

$\Bbbk\text{-dgCat} :=$ the 2-cat of small dg cat^s, dg fun^s, dg nat tr^s.

	dg nat tr	derived
small	$\Bbbk$-dgCat	$\Bbbk$-DGCat
light	$\Bbbk$-dgCAT	$\Bbbk$-DGCAT

Let $\mathcal{A} \in \Bbbk\text{-dgCat}_0 = \Bbbk\text{-DGCat}_0 \ni \mathcal{C}_{dg}(\Bbbk)$

$\mathcal{C}_{dg}(\mathcal{A}) := \Bbbk\text{-DGCat}(\mathcal{A}, \mathcal{C}_{dg}(\Bbbk))$ • obj : right dg $\mathcal{A}$-module $\mathcal{A} \xrightarrow{M} \mathcal{C}_{dg}(\Bbbk)$

$\in \Bbbk\text{-dgCAT}_0 = \Bbbk\text{-DGCAT}_0$ • mor : derived tr.

$\mathcal{C}(\mathcal{A}) := Z^0(\mathcal{C}_{dg}(\mathcal{A}))$: a Frobenius cat

$\mathcal{H}(\mathcal{A}) := H^0(\mathcal{C}_{dg}(\mathcal{A})) = \underline{\mathcal{C}(\mathcal{A})}$: stable cat of $\mathcal{C}(\mathcal{A})$

$$\mathcal{D}(\mathcal{A}) := \mathcal{H}(\mathcal{A})[\mathrm{qis}^{-1}]$$

$$\mathcal{C}_{dg}(\mathcal{A})_0 = \mathcal{C}(\mathcal{A})_0 = \mathcal{H}(\mathcal{A})_0 = \mathcal{D}(\mathcal{A})_0$$

<u>Dfn.</u> Let $M \in \mathcal{C}_{dg}(\mathcal{A})_0$.

$f : M \to N$ in $\mathcal{C}(\mathcal{A})$ is <u>quasi-iso</u> (qis for short) $:\Longleftrightarrow$ $H^n(f) : H^n(M) \to H^n(N)$ <u>iso</u> $\forall n \in \mathbb{Z}$

M : <u>acyclic</u> $:\Longleftrightarrow$ $H^n(M) = 0$, $\forall n \in \mathbb{Z}$.

M : <u>homotopically projective</u> $:\Longleftrightarrow$ $\mathcal{H}(\mathcal{A})(M, A) = 0$, $\forall A$: acyclic
(htp proj) (or <u>cofibrant</u>)

$$\mathcal{H}_p(\mathcal{A}) := \mathrm{full}\{M \in \mathcal{H}(\mathcal{A}) \mid M : \text{htp proj}\}$$

<u>Dfn.</u> $\mathrm{Colax}(I, \Bbbk\text{-dgCat})$: 2-cat

obj : colax functors $I \to \Bbbk\text{-dgCat}$ with suitable 1-mor^s and 2-mor^s.

1-mor : $(F, \phi) : X' \to X$, where $F = (F(i) : X'(i) \to X(i))_{i \in I_0}$ and
dg functor

$\phi = (\phi_a)_{a \in I_1}$
dg nat tr

$$\begin{tikzcd} X'(i) \arrow[r, "F(i)"] \arrow[d, "X'(a)"'] & X(i) \arrow[d, "X(a)"] \arrow[dl, phantom, "\swarrow \phi_a"] \\ X'(j) \arrow[r, "F(j)"'] & X(j) \end{tikzcd}$$

satisfying suitable axioms.

$(a : i \to j$ in $I)$

$$
\begin{tikzcd}
 & \mathcal{C}_{dg}(\Bbbk\text{-dgCat}) \arrow[d, "Z^0"] \arrow[dd, bend left, "H^0"] & \\
\Bbbk\text{-dgCat} \arrow[ur, "\mathcal{C}_{dg}"] \arrow[r, "\mathcal{C}"] \arrow[dr, "\mathcal{H}"] \arrow[ddr, bend right, "\mathcal{D}"'] & \mathcal{C}(\Bbbk\text{-dgCat}) \arrow[d, "st"] & \Bbbk\text{-FRB} \arrow[d, "st"] \arrow[l, hook'] \\
 & \mathcal{H}(\Bbbk\text{-dgCat}) \arrow[d, "\mathbb{L}"] & \Bbbk\text{-TRI} \arrow[l, hook'] \\
 & \Bbbk\text{-}\underline{\mathrm{TRI}}^2 &
\end{tikzcd}
$$

1-mor : $F : \mathcal{C}_{dg}(\mathcal{A}) \to \mathcal{C}_{dg}(\mathcal{B})$ s.t.

$\boxed{F(\mathcal{H}_p(\mathcal{A})) \subseteq \mathcal{H}_p(\mathcal{B})}$

↳ necessary to define $\mathbb{L}$ as a pseudofunctor

$\mathcal{C}_{dg}(X) := \mathcal{C}_{dg} \circ X$

$$
\begin{tikzcd}
 & \Bbbk\text{-dgCat} \arrow[d, Rightarrow, "(F,\phi)"] \arrow[dr, "\mathcal{C}_{dg}"] \arrow[drrrr, bend left, "\mathcal{D}"] & & & & \\
I \arrow[ur, "X'"] \arrow[r, "X"'] & \Bbbk\text{-dgCat} \arrow[r, "\mathcal{C}_{dg}"'] \arrow[rrrr, bend right, "\mathcal{D}"'] & \mathcal{C}_{dg}(\Bbbk\text{-dgCat}) \arrow[r, "Z^0"] \arrow[rr, bend left, "H^0"] & \mathcal{C}(\Bbbk\text{-dgCat}) \arrow[r, "st"] & \mathcal{H}(\Bbbk\text{-dgCat}) \arrow[r, "\mathbb{L}"] & \Bbbk\text{-}\underline{\mathrm{TRI}}^2
\end{tikzcd}
$$

3. Results

Thm Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-dgCat})$. Then (Q):

(1) $\exists (F,\phi) : \mathcal{C}_{dg}(X') \to \mathcal{C}_{dg}(X)$: 1-mor in $\mathrm{Colax}(I, \Bbbk\text{-dgCAT})$

s.t. $\boxed{\forall i \in I_0,\ F(i) \text{ preserves htp projectives,}}$ 0 and

$\mathbb{L}(F,\phi) : \mathcal{D}(X') \to \mathcal{D}(X)$ an eq in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathrm{TRI}}^2)$.

(2) $\exists$ a quasi-eq. 2 $X \xrightarrow{(F',\phi')} T \xrightarrow{(\sigma,\rho)} \mathcal{C}_{dg}(X)$ in $\mathrm{Colax}(I, \Bbbk\text{-dgCAT})$

s.t. T : a tilting colax functor 1 for X

$$
\begin{array}{ccc}
\mathcal{C}_{dg}(X') & \xrightarrow{(F,\phi)} & \mathcal{C}_{dg}(X) \\
 & \Downarrow & \\
\mathcal{D}(X') & \xrightarrow[\mathbb{L}(F,\phi)]{} & \mathcal{D}(X)
\end{array}
$$

$:= \mathbb{L}H^0(F,\phi)$

0. $F(i): X'(i) \to X(i)$ __preserves htp proj__ $:\Leftrightarrow F(i)(\mathcal{H}p(X'(i))_0) \subseteq \mathcal{H}p(X(i))_0$

1. T is a __tilting colax subfunctor__ for X if $\exists$ 1-mor $(\sigma, \rho): T \hookrightarrow \mathcal{C}_{dg}(X)$ with $\sigma(i): T(i) \hookrightarrow \mathcal{C}_{dg}(X(i))$ the inclusion, $T(i)$ is a __tilting dg subcat__ for $X(i)$, $\forall i \in I_0$

1'. A dg subcat T of $\mathcal{C}_{dg}(\mathcal{A})$ is called a __tilting dg subcat for $\mathcal{A}$__ if $\forall M \in T$ is compact and $\mathrm{Loc}(T_0) = \mathcal{D}(\mathcal{A})$.

the smallest localizing subcat of $\mathcal{D}(\mathcal{A})$ containing T_0

Note: no extension vanishing conditions. ($\Rightarrow$ tilting = silting.)

(2) $(F', \phi') : \mathcal{X} \to \mathcal{J}$ is quasi-eq $:\Longleftrightarrow \begin{cases} \forall i \in I_0,\ F(i) : \text{quasi-eq} & \text{— (2)'} \\ \forall a \in I_1,\ \phi(a) : \text{2-qis} & \text{— (2)''} \end{cases}$

(2)' A dg fun $F : \mathcal{A} \to \mathcal{B}$ is quasi-eq $:\Longleftrightarrow \begin{cases} \forall n \in \mathbb{Z},\ H^nF : H^n\mathcal{A} \to H^n\mathcal{B} \text{ fully faith} \\ H^0F : H^0\mathcal{A} \to H^0\mathcal{B} \text{ dense} \end{cases}$

(2)'' A dg nat. tr $\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "E"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$ is 2-qis $:\Longleftrightarrow$ $\begin{tikzcd} \mathcal{D}(\mathcal{A}) \arrow[r, bend left, "-\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} \overline{E}"] \arrow[r, bend right, "-\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} \overline{F}"'] \arrow[r, phantom, "\Downarrow -\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} \overline{\alpha}"] & \mathcal{D}(\mathcal{B}) \end{tikzcd}$

$-\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} \overline{\alpha}$: iso

$\mathcal{C}_{dg}$ $\qquad$ $\mathbb{L}$

$$\begin{tikzcd} \mathcal{C}_{dg}(\mathcal{A}) \arrow[r, bend left, "-\otimes_{\mathcal{A}} \overline{E}"] \arrow[r, bend right, "-\otimes_{\mathcal{A}} \overline{F}"'] \arrow[r, phantom, "\Downarrow -\otimes_{\mathcal{A}} \overline{\alpha}"] & \mathcal{C}_{dg}(\mathcal{B}) \end{tikzcd}$$

${}_{\mathcal{A}}\overline{E}_{\mathcal{B}} := \mathcal{B}(-, E(?))$: bimodule

$\overline{\alpha} := \mathcal{B}(-, \alpha_{(?)})$: a mor of bimodules

Dfn • X' <u>is standardly derived equivalent to</u> X ($X' \overset{sd}{\rightsquigarrow} X$)

$:\Longleftrightarrow$ (1) and/or (2) above holds.

Rmk • $X \overset{sd}{\rightsquigarrow} X$, $X \overset{sd}{\rightsquigarrow} X'$, $X' \overset{sd}{\rightsquigarrow} X'' \Longrightarrow X \overset{sd}{\rightsquigarrow} X''$

• Not clear $X \overset{sd}{\rightsquigarrow} X' \Longrightarrow X' \overset{sd}{\rightsquigarrow} X$

Dfn • X' <u>and</u> X are <u>standardly derived eq</u> ($X \overset{sd}{\sim} X'$)

$:\Longleftrightarrow \exists\, X' = X_0, X_1, \dots, X_n = X$ st. $X_0 \overset{sd}{\rightsquigarrow} X_1 \overset{sd}{\leftsquigarrow} X_2 \overset{sd}{\rightsquigarrow} \cdots \overset{sd}{\rightsquigarrow} X_n$

preserving htp proj naturally

Rmk $(F, \phi) : \mathcal{C}_{dg}(X') \longrightarrow \mathcal{C}_{dg}(X)$ 1-mor st. $\mathbb{L}(F, \phi) : \mathcal{D}(X') \to \mathcal{D}(X)$ eq.

$\Longrightarrow \exists \big(F'(i) : \mathcal{C}_{dg}(X') \longrightarrow \mathcal{C}_{dg}(X)\big)_{i \in I_0}$ st. $\mathbb{L}F'(i) \cong \mathbb{L}F(i)$, $\boxed{F'(i) \text{ preserves htp proj}}$ $\forall i \in I_0$

$\exists\, \psi$ st. $(\mathbb{L}F', \psi) : \mathcal{D}(X') \longrightarrow \mathcal{D}(X)$ is an eq in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathrm{TRI}}^2)$

But not clear whether $\exists\, \psi'$ st. $(F', \psi') : \mathcal{C}_{dg}(X') \to \mathcal{C}_{dg}(X)$ 1-mor

in $\mathrm{Colax}(I, \Bbbk\text{-dgCAT})$ st. $(\mathbb{L}F', \mathbb{L}\psi') : \mathcal{D}(X') \to \mathcal{D}(X)$ is an eq.

Rmk We do not need k-flatness condition on X.

It is possible to remove k-flatness assumption also from Keller's Thm for dg cat^s.

Cor Let $\mathcal{A}, \mathcal{A}'$: small dg cat^s. (Q)

(1) $\exists F : \mathcal{C}_{dg}(\mathcal{A}') \to \mathcal{C}_{dg}(\mathcal{A})$ a dg fun st. $\mathbb{L}F : \mathcal{D}(\mathcal{A}') \to \mathcal{D}(\mathcal{A})$ a tri eq.

(2) $\exists {}_{\mathcal{A}'}U_{\mathcal{A}}$: a bimodule st. $- \overset{\mathbb{L}}{\otimes}_{\mathcal{A}'} U : \mathcal{D}(\mathcal{A}') \to \mathcal{D}(\mathcal{A})$ a tri eq.

(3) $\exists$ a tri eq $\mathcal{D}(\mathcal{A}') \to \mathcal{D}(\mathcal{A})$

(4) $\mathcal{A}'$ is quasi-eq to a tilting dg subcat for $\mathcal{A}$

(5) $\exists F : \mathcal{C}_{dg}(\mathcal{A}') \to \mathcal{C}_{dg}(\mathcal{A})$ a dg fun $\boxed{\text{preserving htp proj}^s}$ st

$\mathbb{L}F : \mathcal{D}(\mathcal{A}') \to \mathcal{D}(\mathcal{A})$ is a tri eq.

Thm 2 Let $X, X' \in \mathrm{Colax}(I, k\text{-dgCat})$.

If X' is quasi-eq to a tilting colax functor for X, then $\int X' \overset{\mathrm{der}}{\sim} \int X$.

<u>Exm</u> In the case that $I = G$ a group, we give an example, for which Thm 2 is applied. $G = \langle g \rangle \cong \mathbb{Z}/(3)$.

- (Q, W): $Q :=$

$$\begin{tikzcd} & & 1 \arrow[dl, "a_1"'] & & \\ & 2 \arrow[dl, "a_2"'] & & 6 \arrow[ul, "a_6"'] & \\ 3 \arrow[rr, "a_3"'] & & 4 \arrow[rr, "a_4"'] & & 5 \arrow[ul, "a_5"'] \end{tikzcd}$$

, $W := a_5 a_4 a_3 a_2 a_1 a_6$

- $(Q', W') := \mu_5 \circ \mu_3 \circ \mu_1 (Q, W)$

$Q' =$

$$\begin{tikzcd} & & 1 \arrow[dr, "a_6^*"] & & \\ & 2 \arrow[ur, "a_1^*"] \arrow[dr, "{[a_3 a_2]}"] & & 6 \arrow[ll, "{[a_1 a_6]}"'] \arrow[dr, "a_5^*"] & \\ 3 \arrow[ur, "a_2^*"] & & 4 \arrow[ll, "a_3^*"] \arrow[ur, "{[a_5 a_4]}"] & & 5 \arrow[ll, "a_4^*"] \end{tikzcd}$$

, $W' = [a_1 a_6] a_6^* a_1^* + [a_3 a_2] a_2^* a_3^* + [a_5 a_4] a_4^* a_5^* + [a_5 a_4][a_3 a_2][a_1 a_6]$

- Define an action of g on (Q, W) by $\begin{cases} i \mapsto i-2 \\ a_i \mapsto a_{i-2} \end{cases}$ (mod 6) $1 \le i \le 6$

- $(Q_G, W_G) := (G, W)/G$

$$Q_G = \begin{tikzcd} 1 \arrow[r, bend right, "\alpha"'] & 2 \arrow[l, bend right, "\beta"'] \end{tikzcd}, \quad W_G = (\beta\alpha)^3$$

- Define an action of g on (Q', W') by $\begin{cases} i \mapsto i-2 \\ a_i^* \mapsto a_{i-2}^*, \ [a_i a_{i+5}] \mapsto [a_{i-2} a_{i+3}] \end{cases}$

- $(Q'_G, W'_G) := (G', W')/G$

$$Q'_G = \begin{tikzcd} 1 \arrow[r, bend right, "\alpha'"'] & 2 \arrow[l, bend right, "\beta'"'] \arrow[loop right, "\gamma"] \end{tikzcd}, \quad W'_G = 3\gamma\beta\alpha + \gamma^3$$

- $\Gamma := \hat{\Gamma}(Q, W)$, $\Gamma' := \hat{\Gamma}(Q', W')$ complete Ginzburg dg algebras (cat')

$G \overset{X}{\curvearrowright} \Gamma$, $G \overset{X'}{\curvearrowright} \Gamma'$

$$\begin{tikzcd} & \Gamma \arrow[r, bend left, "\text{mutation}"] \arrow[d] & \Gamma^{(1)} \arrow[l, "3d"'] \arrow[r, bend left] & \Gamma^{(2)} \arrow[l, "3d"'] \arrow[r, bend left] \arrow[d, phantom, "\Downarrow \text{Thm 2}"] & \Gamma' \arrow[l, "3d"'] \arrow[d] & \\ \hat{\Gamma}(Q_G, W_G) \arrow[r, phantom, "\overset{\text{Morita}}{\sim}"] & \Gamma/G \arrow[rrr, phantom, "\overset{\text{der}}{\sim}"] & & & \Gamma'/G \arrow[r, phantom, "\overset{\text{Morita}}{\sim}"] & \hat{\Gamma}(Q'_G, W'_G) \end{tikzcd}$$

by Keller-Yang Thm 3.2 in [Der eq^s from mut^s of QP]

$\Gamma/G = \int X$, $\Gamma'/G = \int X'$

~~mutation~~

not possible to apply [K-Y, Thm 3.2]

$$\boxed{X' \xrightarrow{q\text{-eq}} T \xhookrightarrow{\text{tilting}} \mathcal{C}_{dg}(\Gamma)}$$

$$
\begin{tikzcd}
\mathcal{H}(A) \arrow[rr, "F"] & & \mathcal{H}(B) \arrow[dd, bend left=60, "Q_B"] \\
\mathcal{H}_p(A) \arrow[u, hook] \arrow[rr, dashed, "F'"] & & \mathcal{H}_p(B) \arrow[u, hook] \\
\mathcal{D}(A) \arrow[u, "P_A"] \arrow[rr, dashed, "\mathbb{L}F"'] & & \mathcal{D}(B) \arrow[u, "\iota_{P_B}"']
\end{tikzcd}
$$