Connectedness of quasi-hereditary structures

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Setting

K : an algebraically closed field
A : a finite dimensional K-algebra with n simple modules, up to isomorphism
$\mathrm{mod}\,A$: the category of finitely generated right A-modules
$S(1), \ldots, S(n)$: a complete set of pairwise non-isomorphic simple A-modules
$P(1), \ldots, P(n)$: projective covers of simple A-modules
$\Lambda = \{1, \ldots, n\}$: the set of labels of simple A-modules
$\mathfrak{S}_n$: the symmetric group on n letters
e : the unit of $\mathfrak{S}_n$
σ_i : the adjacent transposition $(i, i+1)$ for $1 \leq i \leq n-1$
tr_i^σ : the endofunctor of $\mathrm{mod}\,A$ with $\mathrm{tr}_i^\sigma(M) = \sum\{\mathrm{Im}\,\phi \mid \phi : P(j) \to M, \sigma(j) \geq \sigma(i)\}$

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A quasi-hereditary algebra is defined by a pair of an algebra and a total order on the set Λ. On orders for a quasi-hereditary algebra, Dlab and Ringel showed the following.

Theorem [Dlab-Ringel '89]

Let A be a finite dimensional algebra. Then A is hereditary if and only if it is quasi-hereditary for all total orders.

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For an algebra, we call an order inducing a quasi-hereditary algebra a **quasi-hereditary structure**. In this talk, we prove the connectedness of any two permutations giving quasi-hereditary structures.

Main theorem (Theorem 2) [Goto]

Any two permutations giving quasi-hereditary structures are connected.

Definitions

1. Let $\sigma \in \mathfrak{S}_n$. For $i \in \Lambda$, the A-module $\Delta^\sigma(i)$, called the **standard module**, is defined by the maximal factor module of $P(i)$ having only composition factors $S(j)$ with $\sigma(j) \leq \sigma(i)$. Moreover we will write the set $\{\Delta^\sigma(1), \dots, \Delta^\sigma(n)\}$ by Δ^σ.

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2. Dually, we define the **costandard module** $\nabla^\sigma(i)$ and denote the set $\{\nabla^\sigma(1), \dots, \nabla^\sigma(n)\}$ by ∇^σ.
3. We say that an A-module M has a Δ^σ**-filtration** (resp. a ∇^σ**-filtration**) if there is a sequence of submodules $0 = M_{m+1} \subset \cdots \subset M_2 \subset M_1 = M$ such that for each $1 \leq k \leq m$, $M_k/M_{k+1} \cong \Delta^\sigma(j)$ (resp. $M_k/M_{k+1} \cong \nabla^\sigma(j)$) for some $j \in \Lambda$.

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4. A pair (A, σ) is said to be a **quasi-hereditary algebra** provided that the following conditions are satisfied:
   - $[\Delta^\sigma(i) : S(i)] = 1$ for all $i \in \Lambda$.
   - A_A has a Δ^σ-filtration.

   In this case, we say that the permutation σ gives a **quasi-hereditary structure** of A.

We will abbreviate Δ^e and ∇^e as Δ and ∇, respectively.

Example

$$A = K\left(\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}\right)/\langle \alpha\gamma\delta - \beta\delta \rangle.$$

All indecomposable projective modules are of the form

$$P(1) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 3 & & \\ & 4 & \end{smallmatrix},\ P(2) = \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix},\ P(3) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix},\ P(4) = \begin{smallmatrix} 4 \end{smallmatrix}.$$

Then standard modules with respect to e are simple modules and (A, e) is a quasi-hereditary algebra. On the other hand, the standard modules Δ^{σ_3} are as follows.

$$\Delta^{\sigma_3}(1) = \begin{smallmatrix} 1 \end{smallmatrix},\ \Delta^{\sigma_3}(2) = \begin{smallmatrix} 2 \end{smallmatrix},\ \Delta^{\sigma_3}(4) = \begin{smallmatrix} 4 \end{smallmatrix},\ \Delta^{\sigma_3}(3) = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}.$$

Then $P(1)$ does not have Δ^{σ_3}-filtrations, hence (A, σ_3) is "not" quasi-hereditary.

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Then we can characterize the twistability by some conditions on standard or costandard modules.

$$\text{Put } H_i = \dim \mathrm{Hom}(\Delta(i), \Delta(i+1)), E_i = \dim \mathrm{Ext}^1(\Delta(i), \Delta(i+1)),$$
$$\overline{H_i} = \dim \mathrm{Hom}(\nabla(i+1), \nabla(i)), \overline{E_i} = \dim \mathrm{Ext}^1(\nabla(i+1), \nabla(i)).$$

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Theorem 1 [G]

A quasi-hereditary algebra (A, e) is $i\mathrm{th}$-twistable if and only if one of the following conditions holds:

$(\mathcal{E}_i)$ $E_i = 0$ and $\Delta(i+1)$ has a submodule isomorphic to $\Delta(i)^{H_i}$.

$(\overline{\mathcal{E}_i})$ $\overline{E_i} = 0$ and $\nabla(i+1)$ has a factor module isomorphic to $\nabla(i)^{\overline{H_i}}$.

Moreover, if a quasi-hereditary algebra (A, e) satisfies $E_i = \overline{E_i} = 0$, then (A, σ_i) is also quasi-hereditary with $\Delta^{\sigma_i} = \Delta$ and $\nabla^{\sigma_i} = \nabla$.

Lemma 1.1

Let (A, e) be a quasi-hereditary algebra. Then the followings are equivalent.

1. $[\Delta^{\sigma_i}(j) : S(j)] = 1$ for all $j \in \Lambda$.
2. $E_i H_i = 0$.

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Lemma 1.2 [Mazorchuk–Ovsienko '05]

Let (A, e) be a quasi-hereditary algebra. Then the followings hold.

- $\mathrm{Hom}_A(\Delta(i), \Delta(i+1)) \cong \mathrm{Ext}^1_A(\nabla(i+1), \nabla(i))^*$.
- $\mathrm{Ext}^1_A(\Delta(i), \Delta(i+1)) \cong \mathrm{Hom}_A(\nabla(i+1), \nabla(i))^*$.

Here, $-^*$ is the standard K-dual. In particular, we have $H_i = \overline{E_i}$ and $E_i = \overline{H_i}$.

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Lemma 1.3 [Ágoston–Happel–Lukács–Unger '00]

Assume that we have $[\Delta(j) : S(j)] = 1$ for all $j \in \Lambda$.
Then A_A has a Δ-filtration iff $({}_A A)^*$ has a ∇-filtration.

Proof of Theorem 1

Assume that (A, e) is $i\mathrm{th}$-twistable. Moreover we first assume $E_i = 0$. Then $\Delta^{\sigma_i}(j) = \Delta(j)$ for $j \neq i+1$. Since $P(i+1)$ has a Δ^{σ_i}-filtration, so does $\Delta(i+1)$. Hence we have the exact sequence

$$0 \to \Delta^{\sigma_i}(i)^{H_i} \to \Delta(i+1) \to \Delta^{\sigma_i}(i+1) \to 0,$$

that is, the condition $(\mathcal{E}_i)$ holds. On the other hand, assume $H_i = 0$ by Lemma 1.1. Then $\overline{E_i} = 0$ by Lemma 1.2, and hence we can dually show that $(\overline{\mathcal{E}_i})$ holds.

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Conversely, if $(\mathcal{E}_i)$ holds, then $\Delta^{\sigma_i}(j) = \Delta(j)$ for $j \neq i+1$ and there is an exact sequence

$$0 \to \Delta(i)^{H_i} \to \Delta(i+1) \to \Delta^{\sigma_i}(i+1) \to 0.$$

Thus $\Delta(i+1)$, and hence A_A, has a Δ^{σ_i}-filtration. Similarly $(\overline{\mathcal{E}_i})$ implies ${}_A A^*$ has a ∇^{σ_i}-filtration, and hence A_A has a Δ^{σ_i}-filtration by Lemma 1.3.

Example

$$A = K\left(\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd} \right) / \langle \alpha\gamma\delta - \beta\delta \rangle.$$

All permutations giving quasi-hereditary structures of A are as follows. Write $\sigma_{(123121)} = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, for instance.

$$\begin{tikzcd}
 & \sigma_{(12321)} \arrow[r, dash, "\sigma_1"] & \sigma_{(2321)} \arrow[dr, dash, "\sigma_2"] & & & & \\
\sigma_{(123121)} \arrow[ur, dash, "\sigma_2"] \arrow[r, dash, "\sigma_1"] \arrow[dr, dash, "\sigma_3"] & \sigma_{(23121)} \arrow[r, dash, "\sigma_2"] \arrow[dr, dash, "\sigma_3"] & \sigma_{(3121)} \arrow[r, dash, "\sigma_1"] \arrow[dr, dash, "\sigma_3"] & \sigma_{(321)} \arrow[r, dash, "\sigma_3"] & \sigma_{(21)} \arrow[r, dash, "\sigma_2"] & \sigma_{(1)} \arrow[r, dash, "\sigma_1"] & e \\
 & \sigma_{(12312)} \arrow[r, dash, "\sigma_1"] & \sigma_{(2312)} \arrow[dr, dash, "\sigma_2"] & \sigma_{(121)} \arrow[ur, dash, "\sigma_1"] \arrow[r, dash, "\sigma_2"] & \sigma_{(12)} \arrow[r, dash, "\sigma_1"] & \sigma_{(2)} \arrow[ur, dash, "\sigma_2"] & \\
 & & & \sigma_{(312)} \arrow[ur, dash, "\sigma_3"] \arrow[r, dash, "\sigma_1"] & \sigma_{(32)} \arrow[ur, dash, "\sigma_3"] & &
\end{tikzcd}$$

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In general, can we get all permutations giving quasi-hereditary structures from one, by checking repeatedly whether each quasi-hereditary algebra is ith-twistable?

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Definition

Let $\sigma, \tau \in \mathfrak{S}_n$ give quasi-hereditary structures. Then σ and τ are said to be **connected** if there is a decomposition

$$\tau\sigma^{-1} = \sigma_{i_l} \cdots \sigma_{i_1}$$

into the products of adjacent transpositions such that all $\sigma_{i_k} \cdots \sigma_{i_1}\sigma$ also give quasi-hereditary structures for $1 \leq k \leq l$.

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Theorem 2 [G]

Any two permutations giving quasi-hereditary structures are connected.

Lemma 2.1

Let e, σ give quasi-hereditary structures with $e \neq \sigma$. Then for the minimum element $i \in \Lambda$ satisfying $\sigma(i+1) < \sigma(i)$, it holds that $E_i H_i = 0$.

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For such $i = i_1$, we claim the following.

Proposition 2.2

Let e, σ give quasi-hereditary structures. Then there is a minimal decomposition $\sigma = \sigma_{i_l} \cdots \sigma_{i_1}$ into adjacent transpositions such that σ_{i_1} gives a quasi-hereditary structure.

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Inductively applying Proposition 2.2 on the length of a minimal decomposition of σ into adjacent transpositions, then we get the following.

Lemma 2.3

Let e, σ give quasi-hereditary structures. Then they are connected.

Sketch of the proof of Proposition 2.2

Take i the minimum element satisfying $\sigma(i+1) < \sigma(i)$. First, we assume that $E_i = 0$. If $E_i \neq 0$, then $H_i = 0$ by Lemma 2.1, and we can show this statement dually. Put $i_{\sigma+} = \sigma^{-1}(\sigma(i)+1)$.

1. There are two canonical epimorphisms p and q.
2.
3.
4.

$$\Delta^\sigma(i)^{H_i} \xrightarrow{q} \Delta(i)^{H_i} \longrightarrow 0$$

$$P(i+1)/\operatorname{tr}^\sigma_{i_{\sigma+}}(P(i+1)) \xrightarrow{p} \Delta(i+1) \longrightarrow 0$$

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1. There are two canonical epimorphisms p and q.
2. Consider two exact sequences.
3.
4.

$$0 \longrightarrow \operatorname{tr}^e_{i+2}(\Delta^\sigma(i)^{H_i}) \longrightarrow \Delta^\sigma(i)^{H_i} \xrightarrow{q} \Delta(i)^{H_i} \longrightarrow 0$$

$$0 \longrightarrow \operatorname{tr}^e_{i+2}\left(P(i+1)/\operatorname{tr}^\sigma_{i_{\sigma+}}(P(i+1))\right) \longrightarrow P(i+1)/\operatorname{tr}^\sigma_{i_{\sigma+}}(P(i+1)) \xrightarrow{p} \Delta(i+1) \longrightarrow 0$$

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1. There are two canonical epimorphisms p and q.
2. Consider two exact sequences.
3. There is a monomorphism f.
4.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{tr}^e_{i+2}(\Delta^\sigma(i)^{H_i}) \arrow[r] \arrow[d, "\mathrm{tr}^e_{i+2}(f)"] & \Delta^\sigma(i)^{H_i} \arrow[r, "q"] \arrow[d, "f"] & \Delta(i)^{H_i} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{tr}^e_{i+2}\left(P(i+1)/\,\mathrm{tr}^\sigma_{i_{\sigma+}}(P(i+1))\right) \arrow[r] & P(i+1)/\,\mathrm{tr}^\sigma_{i_{\sigma+}}(P(i+1)) \arrow[r, "p"] & \Delta(i+1) \arrow[r] & 0
\end{tikzcd}$$

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Take i the minimum element satisfying $\sigma(i+1) < \sigma(i)$. First, we assume that $E_i = 0$. If $E_i \neq 0$, then $H_i = 0$ by Lemma 2.1, and we can show this statement dually. Put $i_{\sigma+} = \sigma^{-1}(\sigma(i)+1)$.

1. There are two canonical epimorphisms p and q.
2. Consider two exact sequences.
3. There is a monomorphism f.
4. We get a map g commuting the diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{tr}^e_{i+2}(\Delta^\sigma(i)^{H_i}) \arrow[r] \arrow[d, "\mathrm{tr}^e_{i+2}(f)"] & \Delta^\sigma(i)^{H_i} \arrow[r, "q"] \arrow[d, "f"] & \Delta(i)^{H_i} \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{tr}^e_{i+2}\left(P(i+1)/\,\mathrm{tr}^\sigma_{i_{\sigma+}}(P(i+1))\right) \arrow[r] & P(i+1)/\,\mathrm{tr}^\sigma_{i_{\sigma+}}(P(i+1)) \arrow[r, "p"] & \Delta(i+1) \arrow[r] & 0
\end{tikzcd}$$

Then $\mathrm{Coker}(\mathrm{tr}^e_{i+2}(f)) \to \mathrm{Coker}(f)$ is injective. Applying the snake lemma to this diagram, we get $\mathrm{Ker}\, g = 0$. Hence $g : \Delta(i)^{H_i} \to \Delta(i+1)$ is an injection i.e. $(\mathcal{E}_i)$ holds.

Proof of Theorem 2

Theorem 2 [G]

Any two permutations giving quasi-hereditary structures are connected.

Assume that for an algebra A, two permutations σ and τ give quasi-hereditary structures of A. Now we consider the algebra $A' = A$ with new labeled simple modules $S'(1), \ldots, S'(n)$ such that $S'(i) = S(\sigma^{-1}(i))$ for all $i \in \Lambda$. Then e and $\tau\sigma^{-1}$ give quasi-hereditary structures of A'. By Lemma 2.3, we have a decomposition

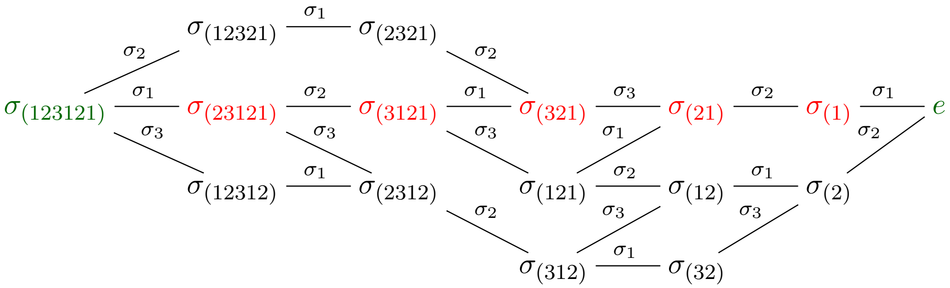
$$\tau\sigma^{-1} = \sigma_{i_l} \cdots \sigma_{i_1}$$

such that all $\sigma_{i_k} \cdots \sigma_{i_1}$ for $1 \leq k \leq l$ give quasi-hereditary structures of A'. This implies that $\sigma_{i_k} \cdots \sigma_{i_1}\sigma$ for $1 \leq k \leq l$ give quasi-hereditary structures of A, that is, σ and τ are connected with respect to giving quasi-hereditary structures of A.

Example

$$A = K\left(\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}\right)/\langle \alpha\gamma\delta - \beta\delta \rangle.$$

If we know that e and $\sigma_{(123121)}$ give quasi-hereditary structures, then by Lemma 2.1, immediately we obtain other permutations giving quasi-hereditary structures below.



Thank you for your attention.