Covering theory of silting objects

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§ Intro

$K = \overline{K}$: field.
A, B : f.d. K-alg.

Tilting theory

$$\begin{array}{ccc} D^b(\operatorname{mod} A) & \cong & D^b(\operatorname{mod} B) \\ \cup\!\!\mid & & \cup\!\!\mid \\ K^b(\operatorname{proj} A) & \cong & K^b(\operatorname{proj} B) \end{array}$$

${}^{\exists}T \in$: tilting s.t. $\operatorname{End}(T) \cong B$

Silting theory

$${}^{\forall}T \quad \cong \quad \operatorname{per} \operatorname{End}(T)$$

$T \in$: silting

Def $\mathcal{T}$: tri. cat. (K-linear, Hom-finite, KS)

- $\mathrm{tilt}\,\mathcal{T} := \{\text{basic tilting obj.s}\}/\cong$

  $\cap$

- $\mathrm{silt}\,\mathcal{T} := \{\text{basic silting obj.s}\}/\cong$

Fact $\mathrm{tilt}\,\mathcal{T}$ & $\mathrm{silt}\,\mathcal{T}$ are posets.

Motivation

I want to clarify the whole picture of $\begin{matrix}\mathrm{tilt}\,\mathcal{T} \\ \mathrm{silt}\,\mathcal{T}\end{matrix}$.

Example

$\underset{(\mathrm{tilt})}{\mathrm{silt}\,A} := \underset{(\mathrm{tilt})}{\mathrm{silt}\,K^b}(\mathrm{proj}\,A)$

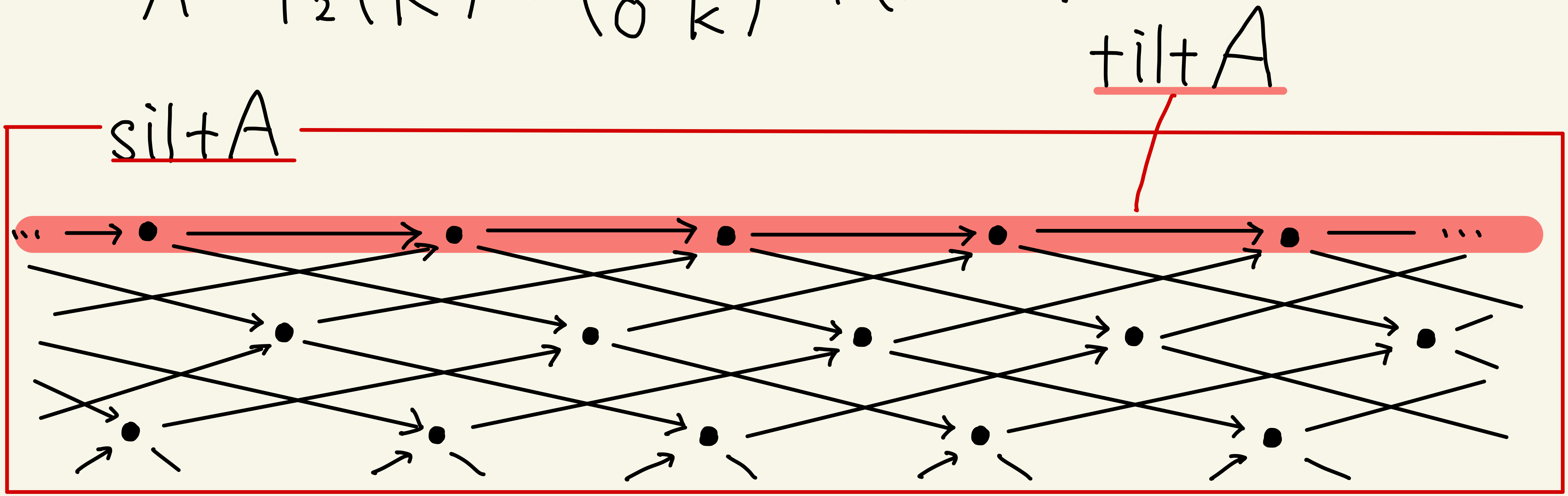
- A : local alg

$\mathrm{silt}\,A = \mathrm{tilt}\,A = \{A[i] \mid i \in \mathbb{Z}\}$

↓ Hasse quiver

$\cdots \to A[-2] \to A[-1] \to A \to A[1] \to A[2] \to \cdots$

- $A = T_2(K) = \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} = K(1 \to 2)$



Thm

- [Iyama-Yang] P: presilt of $\mathcal{T}$

$$\operatorname{silt}_P \mathcal{T} \simeq \operatorname{silt} \mathcal{T}/\operatorname{thick} P$$

- [Aihara-H]

(1) R: local alg.

$$\operatorname{silt} A \simeq \operatorname{silt} A \otimes R$$

(2) $\mathcal{U} \subseteq \mathcal{T}$, $\mathcal{U}$ has a silting obj.

$$\operatorname{silt} \mathcal{U} \subseteq \operatorname{silt} \mathcal{T}$$

__Thm__ [Asashiba, 2011]

Let $\mathcal{C}$: K-linear category with a G-action.

$$\begin{array}{ccc}
\mathrm{tilt}^{G} K^{b}(\mathrm{proj}\,\mathcal{C}) & \longrightarrow & \mathrm{tilt}\, K^{b}(\mathrm{proj}\, \underset{\text{orbit category}}{\mathcal{C}/G}) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\underset{\text{G-stable tilting subcat.}}{\mathcal{M}} & \longmapsto & P_{\bullet}\mathcal{M}
\end{array}$$

In this talk,

$$\mathrm{silt}^{G} K^{b}(\mathrm{proj}\,\mathcal{C}) \longrightarrow \mathrm{silt}\, K^{b}(\mathrm{proj}\,\mathcal{C}/G)$$

Notation

- $K = \overline{K}$: field
- algebra $\Lambda := KQ/I$ $\left(\begin{array}{l} Q \text{ is connected locally finite} \\ \text{but not } \underline{\text{finite}}^{\times}. \end{array}\right.$
- $\mathrm{Aut}(\Lambda)$: the group of automorphisms of Λ
- triangulated cat $\mathcal{T}$: a Krull-Schmidt tri. cat. which is K-linear and Hom-finite

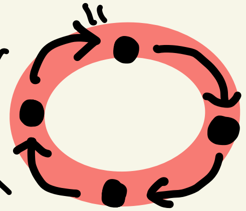
§ Covering theory

[Gabriel, 1981]

$G \leq \mathrm{Aut}(\Lambda)$

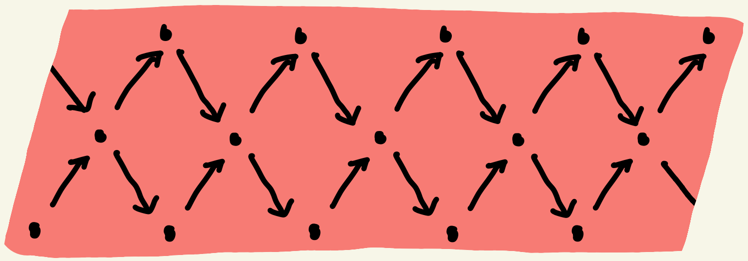
$$G \curvearrowright \Lambda \xrightarrow[\text{functor}]{\text{G-covering}} \Lambda/G : \text{orbit cat.}$$

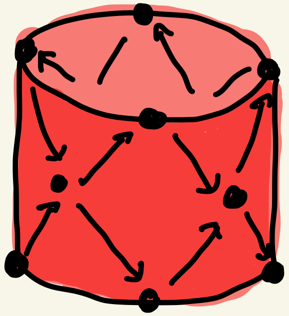
$$\Lambda = K(\cdots \to \bullet \to \bullet \to \bullet \to \bullet \to \cdots)/I$$

$$\Lambda/G = K\Big(\;$$  $$\Big)/I$$

induce

$$\mathrm{mod}\,\Lambda \xrightarrow{\text{G-precovering}} \mathrm{mod}\,\Lambda/G$$

$\cdots$  $\cdots$



$$\Lambda \xrightarrow{\text{G-covering}} \Lambda/G$$

$$\rightsquigarrow$$

[Asashiba, 2011]

$$K^b(\operatorname{proj}\Lambda) \xrightarrow{\text{G-precovering}} K^b(\operatorname{proj}\Lambda/G)$$

[Asashiba-Hafezi-Vahed, 2018]

$$D^b(\operatorname{mod}\Lambda) \longrightarrow D^b(\operatorname{mod}\Lambda/G)$$

$$D_{sg}(\Lambda) \longrightarrow D_{sg}(\Lambda/G)$$

$$D_{sg}(\Lambda)/\underline{\mathrm{Gp}\text{-}\Lambda} \longrightarrow D_{sg}(\Lambda/G)/\underline{\mathrm{Gp}\text{-}\Lambda/G}$$

Classical setting

$\mathcal{C}$: K-linear category, $x, y \in \mathrm{Ob}\,\mathcal{C}$

(1) $\mathcal{C}$: basic $\Leftrightarrow$ ($x \neq y \Rightarrow x \not\cong y$)

(2) $\mathcal{C}$: semiperfect $\Leftrightarrow$ $\mathrm{End}_{\mathcal{C}}(x)$ is a local alg.

(3) G-action is free $\Leftrightarrow$ $\forall g \in G, \forall x \in \mathrm{ind}\,\mathcal{C}$
$\underset{1}{\overset{*}{}}$ $gx \not\cong x$.

(4) G-action is locally bounded
$\Leftrightarrow \{ g \in G \mid \mathrm{Hom}_{\mathcal{C}}(gx, y) \neq 0 \}$ is fin.

Rem $\Lambda = KQ/I \rightsquigarrow \mathcal{C}_{\Lambda}$: $\mathrm{Obj} := Q_0$
K-linear $\qquad \mathrm{Hom}_{\mathcal{C}_\Lambda}(i, j) := e_j \Lambda e_i$

Def (classical)

Let $\mathcal{C}, \mathcal{C}'$: K-linear cat.s

$G \leq \mathrm{Aut}(\mathcal{C})$

$F : \mathcal{C} \to \mathcal{C}'$ is G-precovering

$\overset{\text{def}}{\Leftrightarrow}$ • $^\forall g \in G$, $F = Fg$

• $^\forall x, y \in \mathcal{C}$

$$F_{x,y} : \bigoplus_{g \in G} \mathrm{Hom}_{\mathcal{C}}(gx, y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}'}(Fx, Fy)$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (f_g)_{g \in G} & \longmapsto & \sum_{g \in G} F(f_g) \end{array}$$

- F is G-covering

$\overset{\text{def}}{\iff}$ • F is G-precovering

• F is dence

- $\mathcal{C}/G$: orbit category

$\overset{\text{def}}{\iff}$ $\mathrm{Ob}(\mathcal{C}/G) := \{ Gx \mid x \in \mathcal{C} \}$

$$\mathrm{Hom}_{\mathcal{C}/G}(Gx, Gy) := \left\{ (f_{\beta\alpha})_{\substack{\alpha \in Gx \\ \beta \in Gy}} \in \prod_{\substack{\alpha \in Gx \\ \beta \in Gy}} \mathrm{Hom}_{\mathcal{C}}(x, y) \;\middle|\; g(f_{\beta\alpha}) = f_{g\beta, g\alpha}, \quad \begin{array}{l} \forall g \in G \\ \forall \alpha, \beta \in \mathrm{Ob}\,\mathcal{C} \end{array} \right\}$$

<u>Fact</u> $F:\mathcal{C}\longrightarrow\mathcal{C}'$ is G-covering

$$\Leftrightarrow\quad \begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "P"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathcal{C}' \\ & & \mathcal{C}/G \arrow[u, dashed, "{}^{\exists}\Phi"'] \end{tikzcd} \quad \text{i.e. } F=\Phi P$$

canonical functor

$P:\mathcal{C}\longrightarrow\mathcal{C}/G$ is G-covering

<u>Thm</u>[Asashiba] $P:\Lambda\longrightarrow\Lambda/G$ induces

- $P_\bullet : \operatorname{mod}\Lambda\longrightarrow\operatorname{mod}\Lambda/G$: G-precovering
- $P_\bullet : K^b(\operatorname{proj}\Lambda)\longrightarrow K^b(\operatorname{proj}\Lambda/G)$: G-precovering.

Example

- $\Lambda := K\left(\cdots \longrightarrow -1 \xrightarrow{a_{-1}} 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots\right) / \mathrm{rad}^3$

$$\mathrm{rad}^3 = (\longrightarrow \cdot \longrightarrow \cdot \longrightarrow) = (a_i \cdot a_{i+1} \cdot a_{i+2})$$

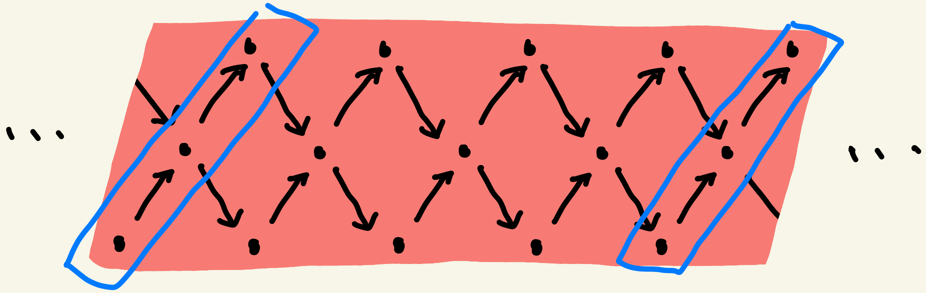
$$\varphi : i \longmapsto i+4$$
$$a_i \longmapsto a_{i+4}$$

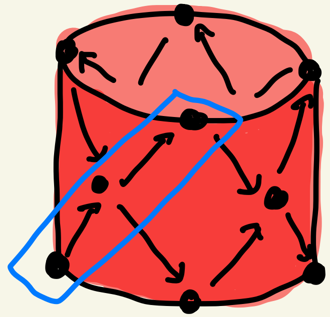
$$G := \langle \varphi \rangle \leq \mathrm{Aut}(\Lambda)$$

$$\Lambda / G = K\left(\begin{tikzcd} & G1 \arrow[dr, bend left] & \\ G0 \arrow[ur, bend left] & & G2 \arrow[dl, bend left] \\ & G3 \arrow[ul, bend left] & \end{tikzcd}\right) \Big/ \mathrm{rad}^3$$

$$\begin{array}{ccc} \Lambda & & \Lambda/G \\ \| & & \| \\ K(\cdots \to \bullet \to \bullet \to \bullet \to \bullet \to \bullet \to \cdots)/I & \xrightarrow{P} & K\left(\begin{tikzcd}[column sep=small, row sep=small] & \bullet \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \arrow[dl] \\ & \bullet \arrow[ul] & \end{tikzcd}\right)/I \end{array}$$

$$\mathrm{mod}\,\Lambda \xrightarrow{\quad P_\bullet \quad} \mathrm{mod}\,\Lambda/G$$





$$\begin{array}{ccc} K^b(\mathrm{proj}\,\Lambda) & \xrightarrow{\quad P_\bullet \quad} & K^b(\mathrm{proj}\,\Lambda/G) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \underbrace{P_i}_{\text{ith ind. proj. module}} & \longmapsto & P_{Gi} \end{array}$$

Silting theory

Def T : obj. or subcat. of $\mathcal{T}$

· T : tilting $\overset{\text{def}}{\iff}$ · $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ $(i \neq 0)$
silting $(i > 0)$

· $\mathrm{thick}\, T = \mathcal{T}$

※ $\mathcal{T}$ has a silting obj.

$\forall \mathcal{M}$: silting subcat, $\exists T$: silting obj.
s.t. $\mathrm{add}\, T = \mathcal{M}$

· $T \geq U \iff \mathrm{Hom}_{\mathcal{T}}(T, U[>0]) = 0$

· $\mathrm{silt}\, \mathcal{T}$:= {the collection of silt. subcat.s}

$\mathcal{T}$ has a silt. obj. $=$ {basic silting obj.s}$/\cong$

Def $T \in \mathrm{silt}\,\mathcal{T}$, $G \leq \mathrm{Aut}(\mathcal{T})$

- T is G-stable $\Leftrightarrow$ $\forall g \in G$
  obj. $gT \cong T$.
  subcat. $\forall M \in T$, $gM \in T$

- $\mathrm{silt}^G\mathcal{T} := \{T \in \mathrm{silt}\,\mathcal{T} \mid T \text{ is G-stable}\}$

- $\mathcal{T}$: G - silting-discrete

$\overset{\text{def}}{\Leftrightarrow}$ $\forall T, U \in \mathrm{silt}^G\mathcal{T}$ with $T \geq U$

$\#\{P \in \mathrm{silt}^G\mathcal{T} \mid T \geq P \geq U\} < \infty$

Main Theorem

<u>Thm</u> [Asashiba, 2011]

Let $\mathscr{C}$: K-linear category with a G-action.

$$\operatorname{tilt}^G K^b(\operatorname{proj} \mathscr{C}) \longrightarrow \operatorname{tilt} K^b(\operatorname{proj} \mathscr{C}/G)$$

<u>Thm</u> [H] Λ : alg (or K-linear cat) with a G-action

(1) $P_\bullet : \operatorname{silt}^G \Lambda \hookrightarrow \operatorname{silt} \Lambda/G$

is an injection preserving partial order.

(2) Λ/G : silting-discrete

$\Rightarrow \Lambda$: G-silting-discrete.

Thm [H] G is torsion-free $\overset{\text{def}}{\Leftrightarrow}$ $|g| < \infty \Rightarrow g = \mathrm{id}_G$

$\Downarrow$

(3) · G acts freely on $\mathrm{ind}\, K^b(\mathrm{proj}\,\Lambda)$

· Λ is G-silting-discrete

$\Rightarrow$ · $P_\bullet : \mathrm{silt}^G \Lambda \xrightarrow{\sim} \mathrm{silt}\, \Lambda/G$ is poset isom.

· Λ/G is silting-discrete.

Cor $H \leq G$

$$\begin{array}{ccc} \mathrm{silt}^G \Lambda & \hookrightarrow & \mathrm{silt}^H \Lambda \\ (3)\ \wr\| & & \wr\| \\ \mathrm{silt}\, \Lambda/G & \hookrightarrow & \mathrm{silt}\, \Lambda/H \end{array}$$

More generally,

$$\mathcal{T} : \text{tri. cat. with a } \underset{\displaystyle G}{\overset{\mathrm{Aut}(\mathcal{T})}{\cup\!\!\!\mid}}\text{-action.}$$

$F : \mathcal{T} \longrightarrow \mathcal{T}'$: G-precovering tri. functor

with $\mathrm{thick}\, F(\mathcal{T}) = \mathcal{T}'$

$\boxed{\begin{array}{l} \underline{\text{Thm}} \text{ [H]} \\ (1)\ F : \mathrm{silt}^G \mathcal{T} \hookrightarrow \mathrm{silt}\, \mathcal{T}' \\ \qquad \text{is an injection preserving partial order.} \\ (2)\ \mathcal{T}' : \text{silting-discrete} \\ \qquad \Rightarrow \mathcal{T} : G\text{-silting-discrete.} \end{array}}$

Thm [H]

(3) · G acts freely on $\operatorname{ind} \mathcal{T}$

· $\mathcal{T}$ is G-silting-discrete

$\Rightarrow$ · $F : \operatorname{silt}^G \mathcal{T} \xrightarrow{\sim} \operatorname{silt} \mathcal{T}'$

is poset isom.

· $\mathcal{T}$ is silting-discrete.

Cor $H \leq G$

$\mathcal{T}$: H-silting-discrete

$\Rightarrow \mathcal{T} : G$-silting-discrete

Example

- $\Lambda := K\left(\cdots \to -1 \xrightarrow{a_{-1}} 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \right) / \mathrm{rad}^3$

$$\psi : \begin{array}{l} i \longmapsto i+1 \\ a_i \longmapsto a_{i+1} \end{array}$$

$G := \langle \psi^2 \rangle$ $\qquad\qquad$ $H := \langle \psi^4 \rangle$

$$\Lambda/G = K\left(\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd} \right) / \mathrm{rad}^3 \qquad \Lambda/H = K\left(\begin{tikzcd} & \bullet \arrow[dr, bend left] & \\ \bullet \arrow[ur, bend left] & & \bullet \arrow[dl, bend left] \\ & \bullet \arrow[ul, bend left] & \end{tikzcd} \right) / \mathrm{rad}^3$$

$$\begin{array}{ccccccc} \mathrm{silt}\,\Lambda/G & \cong & \mathrm{silt}^G \Lambda & \longrightarrow & \mathrm{silt}^H \Lambda & \cong & \mathrm{silt}\,\Lambda/H \\ \rotatebox{90}{$\subseteq$} & & & & & & \rotatebox{90}{$\subseteq$} \\ s\tau\text{-}\mathrm{tilt}\,\Lambda/G & \cong & 2\text{-}\mathrm{silt}^G \Lambda & \longrightarrow & 2\text{-}\mathrm{silt}^H \Lambda & \cong & s\tau\text{-}\mathrm{tilt}\,\Lambda/H \end{array}$$

<u>Fact</u> [Al-Nofayee – Rickard, Aihara, Adachi-Kase]

A : f.d. selfinjective alg.

$\nu := D\mathrm{Hom}_A(-, A)$: Nakayama auto.

$$\mathrm{silt}^{\langle \nu \rangle} A = \mathrm{tilt}\, A$$

not free.

<u>Thm</u> [H] Λ : alg. $G, H \leq \mathrm{Aut}(\Lambda)$

- ${}^{\forall} g \in G, {}^{\forall} h \in H$, $gh = hg$.
- G, H, GH are torsion-free

$$\Rightarrow \underset{\substack{\mathbin{\rotatebox[origin=c]{90}{$\cong$}} \\ \mathrm{silt}\, \Lambda/GH}}{\mathrm{silt}^{GH} \Lambda} \xrightarrow{\sim} \mathrm{silt}^{H} \Lambda/G$$

Example

- $\Lambda := K\left(\cdots \to -1 \xrightarrow{a_{-1}} 0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots\right) / \mathrm{rad}^3$

$$\psi : \begin{array}{rcl} i & \longmapsto & i+1 \\ a_i & \longmapsto & a_{i+1} \end{array}$$

$G := \langle \psi^2 \rangle$ $\qquad\qquad$ $H := \langle \psi^4 \rangle$ $\qquad$ $\nu : i \longmapsto i+2$

$$\Lambda/G = K\left(\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}\right) / \mathrm{rad}^3 \qquad \Lambda/H = K\left(\begin{tikzcd} & 1 \arrow[dr] & \\ 4 \arrow[ur] & & 2 \arrow[dl] \\ & 3 \arrow[ul] & \end{tikzcd}\right) / \mathrm{rad}^3$$

$$\mathrm{silt}\,\Lambda/\underbrace{GH}_{\shortparallel \atop G} = \mathrm{silt}^{GH}\Lambda \cong \mathrm{silt}^{\langle \nu \rangle}\Lambda/H = \mathrm{tilt}\,\Lambda/H$$

Example

$$\Lambda_n = \cdots \quad
\begin{tikzcd}[column sep=small]
(1,-1) \arrow[dr] & & (1,0) \arrow[dr] & & (1,1) \arrow[dr] & & (1,2) \arrow[dr] & & & \\
& (2,-1) \arrow[ur] \arrow[dr] & & (2,0) \arrow[ur] \arrow[dr] & & (2,1) \arrow[ur] \arrow[dr] & & (2,2) \arrow[dr] & & \\
& & (3,-1) \arrow[ur] \arrow[dr, dotted] & & (3,0) \arrow[ur] \arrow[dr, dotted] & & (3,1) \arrow[ur] \arrow[dr, dotted] & & (3,2) \arrow[dr, dotted] & \cdots \\
& & & (n,-1) \arrow[ur] & & (n,0) \arrow[ur] & & (n,1) \arrow[ur] & & (n,2)
\end{tikzcd}$$

mesh relation

$$\varphi : (i,j) \longmapsto (i,j+1)$$

$$A_{n,m} := \Lambda_n / \langle \varphi^m \rangle \qquad \mathcal{A}_n := \Lambda_n / \langle \varphi \rangle$$

[Adachi-Kase, 2022] $2\text{-silt}^{\langle g \rangle} \mathcal{A}_n \cong 2\text{-silt}^{\langle \varphi \rangle} A_{n,m}$

$$\text{silt}\, \mathcal{A}_n \cong \text{silt}^{\langle \varphi \rangle} A_{n,m}$$