Noncommutative conics in Calabi-Yau quantum projective planes II

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Notations

This is a continuation of M. Matsuno's talk, so we will use the same notations as his talk. For example,

- k: an algebraically closed field of characteristic 0. All algebras and vector spaces are over k.
- $k\langle x_1, \ldots, x_d\rangle$ the free algebra.
- $k[x_1, \ldots, x_d]$ the commutative polynomial algebra.

Introduction

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. The noncommutative projective scheme $\mathrm{Proj_{nc}}\, A$ associated to A is called a noncommutative conic in a Calabi-Yau quantum $\mathbb{P}^2$.

Aim

Classify

1. A.
2. $\mathrm{Proj_{nc}}\, A$.
3. $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$: the stable category of the category of maximal Cohen-Macaulay graded right A-modules.
4. E_A: the point variety of A.

Classification of A

There are infinitely many isomorphism classes of Calabi-Yau quantum $\mathbb{P}^2$'s. How many isomorphism classes of A?

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. Then exactly the following two cases occur

(1) A is commutative, and isomorphic to one of the following:

$$k[x,y,z]/(x^2),\ k[x,y,z]/(x^2+y^2),\ k[x,y,z]/(x^2+y^2+z^2).$$

(2) A is not commutative, and isomorphic to:

$$k\langle x,y,z\rangle/(yz+zy+ax^2, zx+xz+by^2, xy+yx+cz^2, \alpha x^2+\beta y^2+\gamma z^2)$$

for some $a,b,c,\alpha,\beta,\gamma \in k$.

For a quadratic algebra $A = T(V)/(R)$, the *quadratic dual* of A is defined as

$$A^! := T(V^*)/(R^\perp),$$

where $V^* = \mathrm{Hom}_k(V, k)$, and

$$R^\perp = \{\theta \in V^* \otimes V^* \mid \theta(x) = 0, \ \forall x \in R\}$$

is the orthogonal complement of R.

Lemma (HMM)

Let $S = \mathcal{A}(E, \sigma)$ be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then $A^! \cong k[X, Y, Z]/(F_1, F_2)$ is a quadratic complete intersection, i.e., $F_1, F_2 \in k[X, Y, Z]_2$ is a regular sequence.

Lemma (Classification of pencils of conics)

There are exactly 6 isomorphism classes of quadratic complete intersections $k[X, Y, Z]/(F_1, F_2)$.

List of quadratic complete intersections $k[X, Y, Z]/(F_1, F_2)$

$k[X,Y,Z]/(X^2, Y^2)$,	$k[X,Y,Z]/(X^2 - YZ, Z^2)$,
$k[X,Y,Z]/(XZ + Y^2, YZ)$,	$k[X,Y,Z]/(X^2 - Y^2, Z^2)$,
$k[X,Y,Z]/(X^2 - YZ, Y^2 - XZ)$,	$k[X,Y,Z]/(X^2 - Y^2, X^2 - Z^2)$.

Lemma

For two quadratic algebras A, A', $A \cong A$ iff $A^! \cong A'^!$.

Corollary (HMM)

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. There are exactly 9 isomorphism classes of A (3 of them are commutative, and 6 of them are not commutative).

List of A

$k[x,y,z]/(x^2)$,	$k[x,y,z]/(x^2+y^2)$,	$k[x,y,z]/(x^2+y^2+z^2)$,
$S^{(0,0,0)}/(x^2)$,	$S^{(0,0,0)}/(x^2+y^2)$,	$S^{(0,0,0)}/(x^2+y^2+z^2)$,
$S^{(1,1,0)}/(x^2)$,	$S^{(1,1,0)}/(3x^2+3y^2+4z^2)$,	$S^{(1,1,0)}/(x^2+y^2-4z^2)$.

Where $S^{(a,b,c)} := k\langle x,y,z\rangle/(yz+zy+ax^2, xz+zx+by^2, xy+yx+cz^2)$.

Classification of E_A and $C(A)$

If S is a d-dim quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$, then $\exists$ a unique regular central element $f^! \in Z(A^!)_2$ s.t. $S^! = A^!/(f^!)$. We define

$$C(A) := A^![(f^!)^{-1}]_0.$$

Theorem (Smith-Van den Bergh (2013))

$\underline{\mathrm{CM}}^{\mathbb{Z}} A \cong \mathscr{D}^b(\operatorname{mod} C(A))$, *where* $\underline{\mathrm{CM}}^{\mathbb{Z}} A$: *stable category of the category of maximal Cohen-Macaulay graded right* A*-modules, and* $\mathscr{D}^b(\operatorname{mod} C(A))$: *the bounded derived category of the category of finitely generated right* $C(A)$*-modules.*

Why E_A

1. For a noncommutative conic A, it is hard to calculate $C(A)$ directly. But we can determine $C(A)$ by calculating E_A.
2. The classification of E_A itself is very interesting.

Theorem (HMM)

Let $S = \mathcal{A}(E, \sigma)$ be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then the following holds:

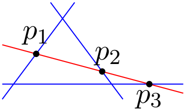
(1) *$C(A)$ is a 4-dim commutative Frobenius algebra.*

(2) *$Z(S)_2 = \{g^2 \mid g \in S_1\}$ (every $f \in Z(S)_2$ is reducible).*

(3) *A satisfies (G1). In fact, if $f = g^2$ where $g \in S_1$, then $\mathcal{P}(A) = (E_A, \sigma_A)$ where*

$$E_A = (E \cap \mathcal{V}(g)) \cup \sigma(E \cap \mathcal{V}(g)), \ \sigma_A = \sigma|_{E_A}.$$

e.g. if $E \cap \mathcal{V}(g)$ be like



then $E_A = \{p_1, p_2, p_3, \sigma(p_1), \sigma(p_2), \sigma(p_3)\}$.

Consider the most interesting case:

$$S = \langle x, y, z\rangle/(yz + zy + \lambda x^2, xz + zx + \lambda y^2, xy + yx + \lambda z^2)$$

where $\lambda \in k$, and $\lambda^3 \neq 0, 1, -8$. Then $S = \mathcal{A}(E, \sigma)$ is a 3-dim Calabi-Yau quantum polynomial algebra where E is an elliptic curve and σ is a translation by a 2-torsion point. Let $f \in Z(S)_2$, and $A = S/(f)$.

What is the possible number $\#(E_A)$?

By the formula $E_A = (E \cap \mathcal{V}(g)) \cup \sigma(E \cap \mathcal{V}(g))$ where $g^2 = f$, and the fact $\#(E \cap \mathcal{V}(g)) \leq 3$ (by Bezout's theorem and the smoothness of a elliptic curve). Then we have cases

$$\begin{aligned}
\#(E \cap \mathcal{V}(g)) = 1 &\Rightarrow \#(E_A) = 2,\\
\#(E \cap \mathcal{V}(g)) = 2 &\Rightarrow \#(E_A) \in \{2, 4\},\\
\#(E \cap \mathcal{V}(g)) = 3 &\Rightarrow \#(E_A) \in \{4, 6\}.
\end{aligned}$$

So we can conclude that $\#(E_A) \in \{2, 4, 6\}$.

Lemma (HMM)

If $S = k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2)$ *is a 3-dim Calabi-Yau quantum polynomial algebra,* $0 \neq f \in Z(S)_2$*, and* $A = S/(f)$*, then* A *is not commutative, and*

$$\operatorname{Spec} C(A) \cong \{(\alpha, \beta, \gamma) \in \mathbb{A}^3 \mid (\alpha x + \beta y + \gamma z)^2 = f \in S\}/\sim$$

where $(\alpha, \beta, \gamma) \sim (-\alpha, -\beta, -\gamma)$*.*

e.g. let $S = k\langle x, y, z\rangle/(xy + yx, xz + zx, yz + zy)$, $f = x^2 \in S_2$.
Then $(\alpha x + \beta y + \gamma z)^2 = f \in S$ implies that

$$\alpha^2 = 1,\ \beta^2 = 0,\ \gamma^2 = 0.$$

Then for $A = S/(f)$, $\operatorname{Spec} C(A) \cong \{(1, 0, 0)\} \subset \mathbb{A}^3$.

Example

If $S = k\langle x, y, z\rangle/(yz + zy, zx + xz, xy + yx)$. Then $S = \mathcal{A}(E, \sigma)$ is a 3-dim Calabi-Yau quantum polynomial algebra, where

$$E = \mathcal{V}(xyz), \text{ and } \begin{cases} \sigma(0, b, c) = (0, b, -c), \\ \sigma(a, 0, c) = (-a, 0, c), \\ \sigma(a, b, 0) = (a, -b, 0). \end{cases}$$

If $f = x^2 + y^2 + z^2 \in Z(S)_2$, and $A = S/(f)$, then

$$(x + y + z)^2 = (x + y - z)^2 = (x - y + z)^2 = (x - y - z)^2 = f$$

in S and $C(A) \cong k^4$, so

$$\operatorname{Spec} C(A) \cong \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1)\} \subset \mathbb{A}^3.$$

Further, if $g = x + y + z$ so that $g^2 = f$, then

$$\begin{aligned} E_A &= (E \cap \mathcal{V}(g)) \cup \sigma(E \cap \mathcal{V}(g)) \\ &= \{(0, 1, -1), (-1, 0, 1), (1, -1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \subset \mathbb{P}^2. \end{aligned}$$

Theorem (HMM)

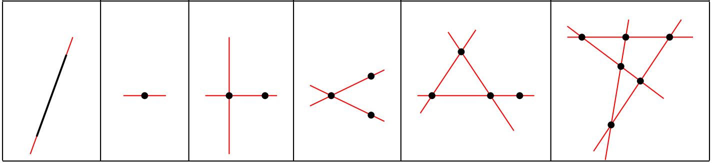
Let S, S' be 3-dim Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$ such that A, A' are not commutative. Then

$$E_A \cong E_{A'} \text{ iff } C(A) \cong C(A').$$

There are exactly 6 isomorphism classes of E_A, so there are exactly 6 isomorphism classes of $C(A)$.

Pictures of E_A when A is not commutative

1 line	1 pt	2 pts	3 pts	4 pts	6 pts



Where "red lines" $\in \operatorname{Spec} C(A)$.

List of $C(A)$ when A is not commutative

$k[u,v]/(u^2,v^2)$,	$k[u]/(u^4)$,	$k[u]/(u^3) \times k$,
$k[u]/(u^2) \times k[u]/(u^2)$,	$k[u]/(u^2) \times k^2$,	k^4.

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$.

Corollary

For any 4-dim commutative Frobenius algebra C, $\exists\ A$ s.t. $C(A) \cong C$.

Corollary

There are exactly 9 isomorphism classes of $C(A)$ (6 of them are commutative, and 3 of them are not commutative).

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$.

List of $C(A)$

$k_{-1}[u,v]/(u^2, v^2)$,	$k_{-1}[u,v]/(u^2, v^2 - 1)$,	$\mathbb{M}_2(k)$,
$k[u,v]/(u^2, v^2)$,	$k[u]/(u^4)$,	$k[u]/(u^3) \times k$,
$k[u]/(u^2) \times k[u]/(u^2)$,	$k[u]/(u^2) \times k^2$,	k^4.

Where $k_{-1}[u,v] = k\langle u,v\rangle/(uv + vu)$.

Corollary

$\underline{\mathrm{CM}}^{\mathbb{Z}}\, A \cong \underline{\mathrm{CM}}^{\mathbb{Z}}\, A'$ *iff* $C(A) \cong C(A')$.

Classification of $\operatorname{Proj}_{\mathrm{nc}} A$

How to check $\operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj}_{\mathrm{nc}} A'$?

Theorem (HMM)

Let S, S' be 3-dim Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, 0 \neq f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$. Then

$$A \cong A' \Rightarrow \operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj}_{\mathrm{nc}} A' \Rightarrow C(A) \cong C(A').$$

Corollary

There are exactly 9 isomorphism classes of noncommutative conics in Calabi-Yau quantum $\mathbb{P}^2$'s.

Classification of smooth conics

Definition

We say that $\operatorname{Proj}_{\mathrm{nc}} A$ is smooth if $\operatorname{gldim}(\operatorname{tails} A) < \infty$.

There are 9 isomorphism classes of noncommutative conics, so how many of them are smooth?

Theorem (Smith-Van den Bergh (2013), Mori-Ueyama (2022))

Let S be a d-dim quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$. Then

$$\operatorname{Proj}_{\mathrm{nc}} A \text{ is smooth iff } C(A) \text{ is semisimple.}$$

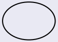
There are exactly two cases $C(A)$ are semisimple, so we have the following result.

Theorem (HMM)

Let S be a 3-dim Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If $\operatorname{Proj_{nc}} A$ is smooth, then exactly the following two cases occur:

(1) (a) $A \cong k[x, y, z]/(x^2 + y^2 + z^2)$ *(A is commutative).*
(b) *f is irreducible.*
(b) $C(A) \cong \mathbb{M}_2(k)$.
(c) $\mathscr{D}^b(\operatorname{tails} A) \cong \mathscr{D}^b(\operatorname{mod} k\widetilde{A}_1)$ *where $k\widetilde{A}_1$ is the path algebra of quiver*

$$1 \rightrightarrows 2 \qquad (\widetilde{A}_1 \text{ type}),$$

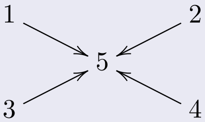
(d) *E_A is the smooth commutative conic:* 

(2) (a) $A \cong S^{(0,0,0)}/(x^2+y^2+z^2)$ *(A is not commutative).*

(b) f *is reducible.*

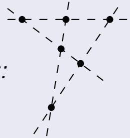
(c) $C(A) \cong k^4$.

(d) $\mathscr{D}^b(\operatorname{tails} A) \cong \mathscr{D}^b(\operatorname{mod} k\widetilde{D}_4)$ *where* $k\widetilde{D}_4$ *is the path algebra of quiver*



($\widetilde{D_4}$ type).

(e) E_A *consists of* 6 *points:*



Where $S^{(0,0,0)} = k\langle x,y,z\rangle/(yz+zy, zx+xz, xy+yx)$.

Summary

Although there are infinitely many Calabi-Yau quantum $\mathbb{P}^2$'s, we have following surprising result.

Theorem

Let $\mathrm{Proj}_{\mathrm{nc}}\, A$, $\mathrm{Proj}_{\mathrm{nc}}\, A'$ *be noncommutative conics in Calabi-Yau quantum* $\mathbb{P}^2$*'s. TFAE.*

(1) $A \cong A'$.

(2) $C(A) \cong C(A')$.

(3) $\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A')$.

(4) $\mathrm{Proj}_{\mathrm{nc}}\, A \cong \mathrm{Proj}_{\mathrm{nc}}\, A'$.

There are exactly 9 *isomorphism classes for each above. Exactly two of* $\mathrm{Proj}_{\mathrm{nc}}\, A$ *are smooth, and exactly one of* $\mathrm{Proj}_{\mathrm{nc}}\, A$ *is irreducible.*

THANK YOU FOR LISTENING!