Symmetric cohomology and symmetric Hochschild cohomology of cocommutative Hopf algebras

Ayako Itaba (Tokyo Univ. of Science)
Yuta Shiba (UBsecure, Inc.)
Katsunori Sanada (Tokyo Univ. of Science)

September 7th, 2022

The 54th Symposium on Ring Theory and Representation Theory @Saitama University

Contents

Previous researches

- G: a group, X: a G-module, $C^n(G,X) = \{f : G^n \to X\}$.
- S_n: the n-th symmetric group.

[Staic, 2009]

- Motivated by topological geometry, [Staic, 2009] defined the symmetric cohomology $\mathrm{HS}^\bullet(G,X)$ of a group G by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard resolution $\mathrm{C}^\bullet(G,X)$ which gives the group cohomology $\mathrm{H}^\bullet(G,X)$.
- Also, [Staic, 2009] studied the injectivity of the canonical map $\mathrm{HS}^\bullet(G,X) \longrightarrow \mathrm{H}^\bullet(G,X)$ induced by the inclusion $\mathrm{CS}^\bullet(G,X) \hookrightarrow \mathrm{C}^\bullet(G,X)$, where $\mathrm{CS}^\bullet(G,X) := \mathrm{C}^\bullet(G,X)^{S_{\bullet+1}}$ is the subcomplex of $\mathrm{C}^\bullet(G,X)$.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{C}^0(G,X) & \longrightarrow & \mathrm{C}^1(G,X) & \longrightarrow & \mathrm{C}^2(G,X) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{H}^\bullet(G,X) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & \mathrm{CS}^0(G,X) & \longrightarrow & \mathrm{CS}^1(G,X) & \longrightarrow & \mathrm{CS}^2(G,X) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HS}^\bullet(G,X)
\end{array}$$

Previous researches

In general, the cohomology of groups can be seen as the cohomology of group algebras.

[Coconet-Todea, 2021]

- Recently, [Coconet-Todea, 2021] defined the symmetric Hochschild cohomology $\mathrm{HHS}^{\bullet}(A, M)$ of twisted group algebras A which is a generalization of group algebras, where M is an A-bimodule.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{C}_{\mathrm{e}}^{0}(A, M) & \longrightarrow & \mathrm{C}_{\mathrm{e}}^{1}(A, M) & \longrightarrow & \mathrm{C}_{\mathrm{e}}^{2}(A, M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HH}^{\bullet}(A, M) \\ & & \hookuparrow & & \hookuparrow & & \hookuparrow & & & & \uparrow \\ 0 & \longrightarrow & \mathrm{CS}_{\mathrm{e}}^{0}(A, M) & \longrightarrow & \mathrm{CS}_{\mathrm{e}}^{1}(A, M) & \longrightarrow & \mathrm{CS}_{\mathrm{e}}^{2}(A, M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HHS}^{\bullet}(A, M) \end{array}$$

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{C}^0(A, M) & \longrightarrow & \mathrm{C}^1(A, M) & \longrightarrow & \mathrm{C}^2(A, M) & \longrightarrow & \cdots \Longrightarrow & \mathrm{H}^\bullet(A, M) \\
 & \uparrow & & \uparrow & & \uparrow & & & \uparrow \\
0 \longrightarrow & \mathrm{CS}^0(A, M) & \longrightarrow & \mathrm{CS}^1(A, M) & \longrightarrow & \mathrm{CS}^2(A, M) & \longrightarrow & \cdots \Longrightarrow & \mathrm{HS}^\bullet(A, M)
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{C}^0_{\mathrm{e}}(A, M) & \longrightarrow & \mathrm{C}^1_{\mathrm{e}}(A, M) & \longrightarrow & \mathrm{C}^2_{\mathrm{e}}(A, M) & \longrightarrow & \cdots \Longrightarrow & \mathrm{HH}^\bullet(A, M) \\
 & \uparrow & & \uparrow & & \uparrow & & & \uparrow \\
0 \longrightarrow & \mathrm{CS}^0_{\mathrm{e}}(A, M) & \longrightarrow & \mathrm{CS}^1_{\mathrm{e}}(A, M) & \longrightarrow & \mathrm{CS}^2_{\mathrm{e}}(A, M) & \longrightarrow & \cdots \Longrightarrow & \mathrm{HHS}^\bullet(A, M)
\end{array}
$$

Aims

We construct the symmetric cohomology (HS) and the symmetric Hochschild cohomology (HHS) for cocommutative Hopf algebras as another generalization of group algebras.

1. We investigate the relationships between the symmetric cohomology (HS) and the symmetric Hochschild cohomology (HHS) for cocommutative Hopf algebras. (Theorem 1)
2. Also, we investigate the relationships between the cohomology (H) and the symmetric cohomology (HS) for cocommutative Hopf algebras. (Theorem 2)

(Cocommutative) Hopf algebras

- k: a field.
- $\otimes = \otimes_k$.

Definition 1

- A: a Hopf algebra over k if A is an k-algebra and a k-coalgebra satisfying

$$\pi \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \varepsilon = \pi \circ (S \otimes \mathrm{id}_A) \circ \Delta,$$

  where the structure morphisms are as follows:
  - $\pi : A \otimes A \to A$: product; $a \otimes b \mapsto ab$,
  - $\eta : k \to A$: unit; $x \mapsto x \cdot 1_A$,
  - $\Delta : A \to A \otimes A$: coproduct, $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$
  - $\varepsilon : A \to k$: counit,
  - $S : A \to A$: antipode (k-linear map). $(A = (A, \pi, \eta, \Delta, \varepsilon))$
- A: cocomm. if $a^{(1)} \otimes a^{(2)} = a^{(2)} \otimes a^{(1)}$ holds.

Examples

1. G: a group, $A = kG$: a group algebra. $\forall g \in G$;
   - coproduct $\Delta(g) := g \otimes g$,
   - counit $\varepsilon(g) := 1$,
   - antipode $S(g) := g^{-1}$,

   $\Longrightarrow$ A: a cocomm. Hopf algebra.
2. $A = k[X]$: a polynomial ring.
   - coproduct $\Delta(X) := 1 \otimes X + X \otimes 1$,
   - counit $\varepsilon(X) := 0$,
   - antipode $S(X) := -X$,

   $\Longrightarrow$ A: a cocomm. Hopf algebra.
3. A: a (cocomm.) Hopf alg. $\Longrightarrow$ A^{op}: a (cocomm.) Hopf alg.
4. A, B: (cocomm.) Hopf alg. $\Longrightarrow$ $A \otimes B$: a (cocomm.) Hopf alg.

   In particular, A: a (cocomm.) Hopf alg
   $\Longrightarrow$ the enveloping alg. $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$: a (cocomm.) Hopf alg.

Modules over Hopf algebras

Definition 2

Let A be a Hopf algebra and M, N left A-modules.

1. For $a \in A$, $m \in M$ and $n \in N$,

   $$a \cdot (m \otimes n) := a^{(1)} m \otimes a^{(2)} n.$$ (Then $M \otimes N$ is a left A-module.)

2. For $a \in A$, $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$,

   $$(a \cdot f)(m) := a^{(1)} f(S(a^{(2)})m).$$ (Then $\mathrm{Hom}_k(M, N)$ is a left A-module.)

3. A submodule ${}^{A}M$ of M is defined by ${}^{A}M := \{m \in M \mid a \cdot m = \varepsilon(a)m\}$, which is called an A-invariant submodule of M. (For a right A-module M, M^A is defined similarly.)
4. Let M an A-bimodule. For $a \in A$ and $m \in M$, $a \cdot m := a^{(1)} m S(a^{(2)})$, which is called a left adjoint action. Using this action, we denote the left A-module by ${}^{\mathrm{ad}}M$. (Similarly, we define a right adjoint action and M^{ad}.)

Cohomology of a Hopf algebra

- A: a Hopf algebra.
- M: a left A-module.
- ${}_A k$; $a \cdot x = \varepsilon(a)x \quad (a \in A,\ x \in k)$.
- $\mathrm{H}^n(A, M) = \mathrm{Ext}^n_A(k, M)$.

The projective resolution of k as left A-modules

- $\widetilde{\mathrm{T}}_n(A) = A^{\otimes n+1}$; $\forall b \in A$,

$$b \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = b^{(1)}a_1 \otimes b^{(2)}a_2 \otimes \cdots \otimes b^{(n+1)}a_{n+1}.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{T}}}} \widetilde{\mathrm{T}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{T}}}} k \longrightarrow 0,$

$$d_n^{\widetilde{\mathrm{T}}}(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i)a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

- $\mathrm{K}^\bullet(A, M) := \mathrm{Hom}_A(\widetilde{\mathrm{T}}_\bullet(A), M)$.

Symmetric cohomology of a cocommutative Hopf algebra

- A: a cocommutative Hopf algebra.
- M: a left A-module.
- S_n: the n-th symmetric group.

Definition 3

Action $\sigma_i = (i,\, i+1) \in S_{n+1}$ on $\mathrm{K}^n(A, M)$; $\forall f \in \mathrm{K}^n(A, M)$, $(1 \leq \forall i \leq n)$

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+1}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}).$$

- $\mathrm{KS}^\bullet(A, M) := \mathrm{K}^\bullet(A, M)^{S_{\bullet+1}} \subset \mathrm{K}^\bullet(A, M)$.

Definition 4

$$\mathrm{HS}^n(A, M) := \mathrm{H}^n(\mathrm{KS}^\bullet(A, M)).$$

Hochschild cohomology of a Hopf algebra

- A: a Hopf algebra.
- M: an A-bimodule $\quad (A^{\mathrm{e}} = A \otimes A^{\mathrm{op}},\ {}_{A^{\mathrm{e}}}M)$.
- $\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, M)$.

The projective resolution of A as A-bimodules

- $\widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) = A^{\otimes n+2}$; $\forall b \otimes c^{\mathrm{op}} \in A^{\mathrm{e}}$,

$$(b \otimes c^{\mathrm{op}}) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2}) = b^{(1)} a_1 \otimes b^{(2)} a_2 \otimes \cdots \otimes b^{(n+2)} a_{n+2} c.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{T}}^{\mathrm{e}}}} \widetilde{\mathrm{T}}^{\mathrm{e}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{T}}^{\mathrm{e}}}} A \longrightarrow 0,$

$$d_n^{\widetilde{\mathrm{T}}^{\mathrm{e}}}(a_1 \otimes \cdots \otimes a_{n+2}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i) a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

- $\mathrm{K}^{\bullet}_{\mathrm{e}}(A, M) := \mathrm{Hom}_{A^{\mathrm{e}}}(\widetilde{\mathrm{T}}^{\mathrm{e}}_{\bullet}(A), M)$.

Symm. Hochschild cohomology of a cocomm. Hopf alg.

- A: a cocommutative Hopf algebra, M: an A-bimodule.
- S_n: the n-th symmetric group.

Definition 5

Action $\sigma_i = (i,\ i+1) \in S_{n+1}$ on $\mathrm{K}_\mathrm{e}^n(A,M)$; $\forall f \in \mathrm{K}_\mathrm{e}^n(A,M)$, $(1 \leq \forall i \leq n)$

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+2}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+2}).$$

- $\mathrm{KS}_\mathrm{e}^\bullet(A,M) := \mathrm{K}_\mathrm{e}^\bullet(A,M)^{S_{\bullet+1}} \subset \mathrm{K}_\mathrm{e}^\bullet(A,M)$.

Definition 6

$$\mathrm{HHS}^n(A,M) := \mathrm{H}^n(\mathrm{KS}_\mathrm{e}^\bullet(A,M)).$$

Aim

1. We investigate the relationships between the symmetric cohomology (HS) and the symmetric Hochschild cohomology (HHS) for cocommutative Hopf algebras. (Theorem 1)

[Eilenberg-MacLane, 1947], [Ginzburg-Kumar, 1993]

Theorem ([Eilenberg-MacLane, 1947])

Let G be a group and X a G-bimodule. Then, for each $n \geq 0$, there is an isomorphism

$$\mathrm{HH}^n(\mathbb{Z}G, X) \cong \mathrm{H}^n(G, {}^{\mathrm{ad}}X)$$

as $\mathbb{Z}$-modules, where ${}^{\mathrm{ad}}X$ is a left G-module by $g \cdot x = gxg^{-1}$ for $g \in G$ and $x \in X$.

- The above theorem is generalized to the case of Hopf algebras by [Ginzburg-Kumar, 1993].
- For cocomutative Hopf algebras, we have the following result which is a symmetric version of the classical results by [Eilenberg-MacLane] and [Ginzburg-Kumar].

Main result 1

Theorem 1 ([I.-Shiba-Sanada, 2022])

Let A be a cocommutative Hopf algebra and M an A-bimodule. Then, for each $n \geq 0$, there is an isomorphism

$$\mathrm{HHS}^n(A, M) \cong \mathrm{HS}^n(A, {}^{\mathrm{ad}}M)$$

as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action, that is, $a \cdot m = a^{(1)} m S(a^{(2)})$ for $m \in {}^{\mathrm{ad}}M$ and $a \in A$.

Corollary 1 ([I.-Shiba-Sanada, 2022])

Let A be a finite dimensional, commutative and cocommutative Hopf algebra. Then, for each $n \geq 0$, there is an isomorphism

$$\mathrm{HHS}^n(A, A) \cong A \otimes \mathrm{HS}^n(A, k)$$

as k-vector spaces.

Second aim

Aim

2. Also, we investigate the relationships between the cohomology (H) and the symmetric cohomology (HS) for cocommutative Hopf algebras. (Theorem 2)

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{C}^0(A, M) & \longrightarrow & \mathrm{C}^1(A, M) & \longrightarrow & \mathrm{C}^2(A, M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{H}^\bullet(A, M) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & \mathrm{CS}^0(A, M) & \longrightarrow & \mathrm{CS}^1(A, M) & \longrightarrow & \mathrm{CS}^2(A, M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HS}^\bullet(A, M)
\end{array}
$$

([Staic, 2009], [Todea, 2015])

- $\mathrm{HS}^0(G, X) \cong \mathrm{H}^0(G, X)$.
- $\mathrm{HS}^1(G, X) \cong \mathrm{H}^1(G, X)$.
- $\mathrm{HS}^2(G, X) \hookrightarrow \mathrm{H}^2(G, X)$.
  - G has no elements of order 2 $\Longrightarrow \mathrm{HS}^2(G, X) \cong \mathrm{H}^2(G, X)$.

Isomorphism as complexes

The resolution of k

- k: a trivial left kS_{n+1}-module; $\tau \cdot x = \varepsilon(\tau)x = x \quad (\tau \in S_{n+1},\ x \in k)$.
- $\widetilde{\mathrm{T}}_n(A)$: a right kS_{n+1}-module; $\forall \sigma_i \in S_{n+1}$, $(1 \leq \forall i \leq n)$

$$(a_1 \otimes \cdots \otimes a_{n+1}) \cdot \sigma_i = -a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}.$$

- $\widetilde{\mathrm{S}}_n(A) := \widetilde{\mathrm{T}}_n(A) \otimes_{kS_{n+1}} k$.
- $\cdots \longrightarrow \widetilde{\mathrm{S}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{S}}}} \cdots \longrightarrow \widetilde{\mathrm{S}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{S}}}} k \longrightarrow 0$,

$$d_n^{\widetilde{\mathrm{S}}}((a_1 \otimes \cdots \otimes a_{n+1}) \otimes_{kS_{n+1}} x) = d_n^{\widetilde{\mathrm{T}}}((a_1 \otimes \cdots \otimes a_{n+1})) \otimes_{kS_n} x.$$

Isomorphism as complexes

$$\mathrm{KS}^\bullet(A, M) \cong \mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M).$$

$$\mathrm{HS}^n(A, M) \cong \mathrm{H}^n(\mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M)).$$

Main result 2

Theorem 2 ([I.-Shiba-Sanada, 2022])

Let A be a cocommutative Hopf algebra. For each $n \geq 1$, if $\mathrm{ch}\, k \nmid n+1$, then $\widetilde{\mathrm{S}}_n(A)$ is projective as a left A-module.
Therefore, if $\mathrm{ch}\, k \nmid (n+1)!$, then, for each $0 \leq m \leq n$, there is an isomorphism

$$\mathrm{H}^m(A, M) \cong \mathrm{HS}^m(A, M)$$

as k-vector spaces.

Remark

- By Theorem 2, if $\mathrm{ch}\, k = 0$, then $\widetilde{\mathrm{S}}_\bullet(A)$ is a projective resolution of k, and hence there is an isomorphism $\mathrm{H}^\bullet(A, M) \cong \mathrm{HS}^\bullet(A, M)$ as k-vector spaces.
- Moreover, by Theorem 1 and Theorem 2, if $\mathrm{ch}\, k = 0$, then there is an isomorphism $\mathrm{H}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HS}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HHS}^\bullet(A, M)$ as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action.

Example

- p: an odd prime number, k: a field of characteristic p.
- C_p: a cyclic group of order p.

Then we calculate the symmetric cohomology of $A = kC_p$.

Proposition 1

Let p be an odd prime number, $\mathrm{ch}\, k = p$ and $A = kC_p$. Then $\widetilde{\mathrm{S}}_n(A)$ is a free A-module with rank $\dfrac{{}_pC_{n+1}}{p}$ for each $1 \leq n \leq p-2$.

- Since $\widetilde{\mathrm{S}}_{p-1}(A)$ is isomorphic to k as a left A-module, the resolution of k is the following exact sequence

$$0 \to k \xrightarrow{d^{\widetilde{\mathrm{S}}}_{p-1}} \widetilde{\mathrm{S}}_{p-2}(A) \to \cdots \to \widetilde{\mathrm{S}}_1(A) \xrightarrow{d^{\widetilde{\mathrm{S}}}_1} \widetilde{\mathrm{S}}_0(A) \xrightarrow{d^{\widetilde{\mathrm{S}}}_0} k \to 0,$$

where $\widetilde{\mathrm{S}}_i(A)$ is a free A-module for each $0 \leq i \leq p-2$.

Example

- This implies that there is an isomorphism

$$\mathrm{H}^n(A, M) \cong \mathrm{HS}^n(A, M)$$

for any left $A = kC_p$-module M and each $0 \leq n \leq p-2$.

- Also, in the case of $n = p-1$, the above isomorphism is obtained by simple calculation.
- Note that the period of the cohomology group $\mathrm{H}^n(A, M)$ of A is 2.

Summarizing the above, we have

$$\mathrm{HS}^n(A, M) \cong \begin{cases} \mathrm{H}^n(A, M) & (0 \leq n \leq p-1), \\ 0 & (p \leq n). \end{cases}$$

Thank you for your attention !

If you have an interest in our talk, please see arXiv:2203.17043.