Noncommutative conics in Calabi-Yau quantum projective planes I

Haigang Hu, Masaki Matsuno, Izuru Mori

Shizuoka University

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Motivations

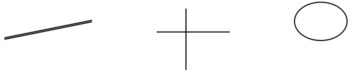
- k: an algebraically closed field of characteristic 0.
- $\mathbb{P}^{d-1}$: the $d-1$ dimensional projective space over k.

Every quadric hypersurface in $\mathbb{P}^{d-1}$ is isomorphic to

$$\operatorname{Proj} k[x_1, \dots, x_d]/(x_1^2 + \cdots + x_j^2)$$

for some $j = 1, \dots, d$.

When $d = 3$, there are exactly 3 isomorphism classes of conics.



Aim

Define and classify noncommutative conics.

Quantum polynomial algebras

Definition ([Artin-Schelter, 1987])

A d-dimensional quantum polynomial algebra is a noetherian connected graded algebra S ($S = \oplus_{i \in \mathbb{N}} S_i, S_0 = k$) such that

1. $\operatorname{gldim} S = d < \infty$,
2. (Gorenstein condition) $\operatorname{Ext}^i_S(k, S) = \begin{cases} k & (i = d), \\ 0 & (i \neq d), \end{cases}$
3. $H_S(t) := \sum_{i=0}^{\infty} (\dim_k S_i) t^i = 1/(1-t)^d$.

Example

- If S is commutative, S: d-dimensional quantum polynomial algebra $\Leftrightarrow S \cong k[x_1, \dots, x_d]$.
- $S := k\langle x, y, z \rangle / (yz + zy, zx + xz, xy + yx)$ is a 3-dimensional quantum polynomial algebra.

Geometric algebras

- V: a finite dimensional vector space over k, $R \subset V \otimes V$.
- $T(V)$: the tensor algebra of V.
- $\mathcal{V}(R) := \{(p,q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p,q) = 0, \forall f \in R\}$.

Definition ([Mori, 2006])

Let $S = T(V)/(R)$ be a quadratic algebra.

1. A geometric pair (E, σ) consistes of a projective scheme $E \subset \mathbb{P}(V^*)$ and an automorphism $\sigma \in \operatorname{Aut} E$.
2. S satisfies (G1) $(\mathcal{P}(A) = (E, \sigma))$ $:\Longleftrightarrow \exists (E, \sigma)$: a geometric pair such that $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$.
3. S satisfies (G2) $(A = \mathcal{A}(E, \sigma))$ $:\Longleftrightarrow \exists (E, \sigma)$: a geometric pair such that $R = \{f \in V \otimes V \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.
4. S: *geometric* $:\Longleftrightarrow$ S satisfies (G1), (G2) and $S = \mathcal{A}(\mathcal{P}(S))$.

Geometric algebras

Example

- $S = k[x_1, \dots, x_d] = \mathcal{A}(\mathbb{P}^{d-1}, \mathrm{id})$ is a geometric algebra.
- $S = k\langle x, y, z\rangle/(yz + zy, zx + xz, xy + yx) = \mathcal{A}(E, \sigma)$ is a geometric algebra where $E = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z) \subset \mathbb{P}^2$ and

$$\begin{cases} \sigma(0, b, c) = (0, b, -c) \\ \sigma(a, 0, c) = (-a, 0, c) \\ \sigma(a, b, 0) = (a, -b, 0). \end{cases}$$

- 3-dimensional Sklyanin algebra

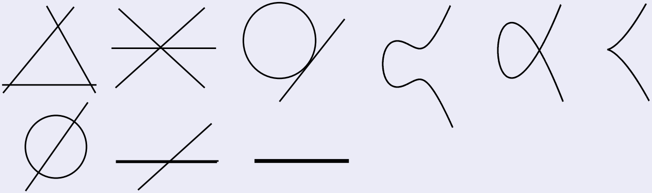
$$S = k\langle x, y, z\rangle/(\alpha yz + \beta zy + \gamma x^2, \alpha zx + \beta xz + \gamma y^2, \alpha xy + \beta yx + \gamma z^2)$$

is a geometric algebra with $\mathcal{P}(S) = (E, \sigma)$ where
  - $E = \mathcal{V}(x^3 + y^3 + z^3 - \lambda xyz) \subset \mathbb{P}^2, \lambda = \frac{\alpha^3+\beta^3+\gamma^3}{\alpha\beta\gamma}$ (elliptic curve),
  - $\sigma(q) := p + q, p = (\alpha, \beta, \gamma)$ (translation).

Geometric algebras

Theorem ([Artin-Tate-Van den Bergh, 1990])

Every 3-dimensional quantum polynomial algebra $S = \mathcal{A}(E, \sigma)$ is geometric where either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a cubic curve.



Twisted superpotentials

Definition

Let $m \in \mathbb{N}^+$. Define a linear map $\phi : V^{\otimes m} \to V^{\otimes m}$ by
$\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) = v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}$.

1. $w \in V^{\otimes m}$ is called superpotential if
$$\phi(w) = w.$$
2. $w \in V^{\otimes m}$ is called twisted superpotential if
$$(\tau \otimes \mathrm{id}^{\otimes m-1})\phi(w) = w$$
for some $\tau \in \mathrm{GL}(V)$.
3. The i-th derivation quotient algebra of $w \in V^{\otimes m}$ is defined by
$$D(w, i) := T(V)/(\partial^i w)$$
where $\partial^i w$ is the "i-th left partial derivatives" of w.

Twisted superpotentials

We fix a basis $\{x, y, z\}$ for V.

Example

- $w = xyz + yzx + zxy - (xzy + yxz + zyx)$ is a superpotential,
$$D(w, 1) = k\langle x, y, z\rangle/(yz - zy, zx - xz, xy - yx) = k[x, y, z].$$
- $w = xyz + yzx + zxy + (xzy + yxz + zyx)$ is a superpotential,
$$D(w, 1) = k\langle x, y, z\rangle/(yz + zy, zx + xz, xy + yx).$$
- $w = \alpha(xyz + yzx + zxy) + \beta(xzy + yxz + zyx) + \gamma(x^3 + y^3 + z^3)$ is a superpotential,
$D(w, 1) =$
$k\langle x, y, z\rangle/(\alpha yz + \beta zy + \gamma x^2, \alpha zx + \beta xz + \gamma y^2, \alpha xy + \beta yx + \gamma z^2)$.

Twisted superpotentials

Theorem ([Dubois-Violette,2007])

For every d-dimensional quantum polynomial algebra S, there exists a unique twisted superpotential w such that $S = D(w, d-2)$.

Theorem ([Mori-Smith,2016])

Let w be a twisted superpotential such that $S = D(w, d-2)$ is a d-dimensional quantum polynomial algebra. Then S is "Calabi-Yau" if and only if w is $(-\mathrm{id})^{d+1}$ twisted superpotential.

S: 3-dimensional quantum polynomial algebra

S is "Calabi-Yau" $\iff$ w : superpotential

Twisted superpotentials

Theorem ([Mori-Smith,2017])

Superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.

(Classification of 3-dimensional Calabi-Yau quantum polynomial algebras)

Theorem ([Itaba-M,2022])

Twisted superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.

(Classification of 3-dimensional quantum polynomial algebras)

Quantum projective spaces

Definition ([Artin-Zhang,1994])

A noncommutative schem is a pair $X = (\operatorname{mod} X, \mathcal{O}_X)$ consisting of a k-linear abelian category $\operatorname{mod} X$ and an object $\mathcal{O}_X \in \operatorname{mod} X$.
Two noncommutative schemes X and Y are isomorphic, denoted by $X \cong Y$, if there exists an equivalence functor $F : \operatorname{mod} X \to \operatorname{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

Example

If X is a commutative noetherian scheme, then we view X as a noncommutative scheme by $X = (\operatorname{coh} X, \mathcal{O}_X)$.

Quantum projective spaces

Definition

Let A be a right noetherian connected graded algebra. The noncommutative projective scheme associated to A is a noncommutative scheme defined by $\operatorname{Proj}_{\mathrm{nc}} A := (\operatorname{tails} A, \pi A)$ where

- $\operatorname{grmod} A$: the category of finitely generated graded right A-module,
- $\operatorname{tors} A$: the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k,
- $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$: the quotient category,
- $\pi : \operatorname{grmod} A \to \operatorname{tails} A$: the quotient functor.

Example

If A is commutative and generated in degree 1 over k, then

$$\operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj} A.$$

Quantum projective spaces

Definition

A quantum $\mathbb{P}^{d-1}$ is a noncommutative projective scheme $\operatorname{Proj}_{\mathrm{nc}} S$ for some d-dimensional quantum polynomial algebra S.

Theorem ([Itaba-M,2022])

For every 3-dimensional quantum polynomial algebra S, there exists a 3-dimensional Calabi-Yau quantum polynomial algebra $\tilde{S}$ such that $\operatorname{Proj}_{\mathrm{nc}} S \cong \operatorname{Proj}_{\mathrm{nc}} \tilde{S}$.

$$\{\text{quantum } \mathbb{P}^2\text{'s}\} = \{\text{“Calabi-Yau” quantum } \mathbb{P}^2\text{'s}\}$$

Noncommutative conics

Definition

A noncommutative quadric hypersurface in a (Calabi-Yau) quantum $\mathbb{P}^{d-1}$ is the noncommutative projective scheme $\operatorname{Proj}_{\mathrm{nc}} S/(f)$ for some d-dimensional (Calabi-Yau) quantum polynomial algebra S and for some regular central element $f \in S_2$.
When $d = 3$, we say that $\operatorname{Proj}_{\mathrm{nc}} S/(f)$ is a noncommutative conic.

Aim

We study and classify homogeneous coordinate algebras $S/(f)$.

Noncommutative conics

Theorem ([Hu,Hu-M-Mori])

Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra. Then $Z(S)_2 \neq 0$ if and only if S is isomorphic to one of the algebras in Table 1.

Table 1

defining relations	$Z(S)_2$
$yz - zy, zx - xz, xy - yx$	S_2
$yz - zy + x^2, zx - xz, xy - yx$	kx^2
$yz + zy, zx + xz, xy + yx$	$kx^2 + ky^2 + kz^2$
$yz + zy + x^2, zx + xz, xy + yx$	$kx^2 + ky^2 + kz^2$
$yz + zy + x^2, zx + xz + y^2, xy + yx$	$kx^2 + ky^2 + kz^2$
$yz + zy + \lambda x^2, zx + xz + \lambda y^2, xy + yx + \lambda z^2$	$kx^2 + ky^2 + kz^2$

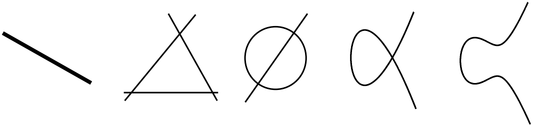
where $\lambda \in k$ such that $\lambda^3 \neq 0, 1, -8$.

Noncommutative conics

Table 2

defining relations	$\lvert\sigma\rvert$	Type
$yz - zy, zx - xz, xy - yx$	1	P
$yz - zy + x^2, zx - xz, xy - yx$	1	TL
$yz + zy, zx + xz, xy + yx$	2	S
$yz + zy + x^2, zx + xz, xy + yx$	2	S'
$yz + zy + x^2, zx + xz + y^2, xy + yx$	2	NC
$yz + zy + \lambda x^2, zx + xz + \lambda y^2, xy + yx + \lambda z^2$	2	EC

where $\lambda \in k$ such that $\lambda^3 \neq 0, 1, -8$. In this case, $E = \mathbb{P}^2$ or



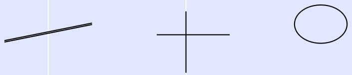
Noncommutative conics

Theorem ([Hu-M-Mori])

Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is commutative, then

1. either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a triple line, and
2. A is isomorphic to one of the following algebras:

$$k[x, y, z]/(x^2), \ \ k[x, y, z]/(x^2 + y^2), \ \ k[x, y, z]/(x^2 + y^2 + z^2).$$



Noncommutative conics

Let

$$\begin{aligned} w^{(a,b,c)} &:= xyz + yzx + zxy + (xzy + yxz + zyx) + (ax^3 + by^3 + cz^3) \\ S^{(a,b,c)} &:= k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2) \\ &= D(w^{(a,b,c)}, 1) \end{aligned}$$

where $(a, b, c) = (0, 0, 0), (1, 0, 0), (1, 1, 0), (\lambda, \lambda, \lambda), \lambda^3 \neq 0, 1, -8$.

Let $\mathrm{Sym}(3)$ be the symmetric group of degree 3 and

$$\mathrm{Sym}^3(V) := \{w \in V^{\otimes 3} \mid \theta \cdot w = w \text{ for all } \theta \in \mathrm{Sym}(3)\}.$$

$\Longrightarrow$ Every $w^{(a,b,c)}$ belongs to $\mathrm{Sym}^3(V)$.

Noncommutative conics

Theorem ([Hu-M-Mori])

Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then

1. $|\sigma| = 2$, and
2. $S = D(w, 1)$ for some $w \in \mathrm{Sym}^3(V)$, and A is isomorphic to

$$S^{(a,b,c)}/(\alpha x^2 + \beta y^2 + \gamma z^2)$$

where $(\alpha, \beta, \gamma) \in \mathbb{P}^2$.

Remark

There are infinitely many isomorphism classes of $S^{(a,b,c)}$.
(How many isomorphism classes of A ?)

Noncommutative conics

Example

Let

$$S^{(0,0,0)} = k\langle x, y, z\rangle/(yz + zy, zx + xz, xy + yx),$$
$$S^{(1,0,0)} = k\langle x, y, z\rangle/(yz + zy + x^2, zx + xz, xy + yx),$$
$$S^{(1,1,0)} = k\langle x, y, z\rangle/(yz + zy + x^2, zx + xz + y^2, xy + yx).$$

Then

1. $S^{(0,0,0)}/(x^2) \cong S^{(1,0,0)}/(x^2)$.
2. $S^{(1,0,0)}/(y^2) \cong S^{(1,1,0)}/(x^2)$.
3. $S^{(0,0,0)}/(x^2 + y^2) \cong S^{(1,0,0)}/(x^2 + y^2) \cong S^{(1,1,0)}/(x^2 + y^2)$.
4. $S^{(0,0,0)}/(x^2 + y^2 + z^2) \cong S^{(1,0,0)}/(y^2 + z^2)$.