Localization of triangulated categories with respect to extension-closed subcategories

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Notation and Conventions

Notation

- a category = an additive category.
- a subcategory = a full and additive subcategory.
- a functor = an additive functor.
- $\mathcal{C}$: triangulated category.
- $\mathcal{N}$: a full subcategory which is closed under $\begin{cases} \text{direct summands} \\ \text{isomorphisms} \\ \text{extensions} \end{cases}$.

Verdier quotient

Definition

A subcategory $\mathcal{N} \subseteq \mathcal{C}$ is *thick* if it satisfies the 2-out-of-3 for triangles, namely, for any triangle

$$A \to B \to C \to A[1]$$

if two of $\{A, B, C\}$ belong to $\mathcal{N}$, then so does the third.

Remark

Note that the above definition is same as the usual one. In particular, a thick subcategory $\mathcal{N}$ is triangulated.

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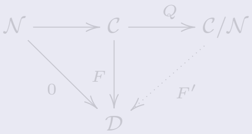
Theorem (Verdier)

For a thick subcategory $\mathcal{N}$, we have a *triangle* functor

$$Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$$

with the universality: For any triangle functor $F : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{N} \subseteq \operatorname{Ker} F$, there uniquely exists a triangle functor $F' : \mathcal{C}/\mathcal{N} \to \mathcal{D}$ which makes the following diagram commutative.



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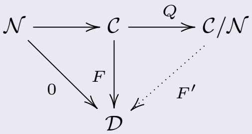

Heart of t-structure

Definition

$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$: a pair of subcategories of $\mathcal{C}$. The pair is called a *t-structure* if it satisfies:

1. $\mathrm{Hom}_{\mathcal{C}}(U, V) = 0$ for any $U \in \mathcal{C}^{\leq 0}$ and $V \in \mathcal{C}^{\geq 1}$;
2. $\mathcal{C}^{\leq 0}[1] \subseteq \mathcal{C}^{\leq 0}$;
3. $\mathcal{C} = \mathcal{C}^{\leq 0} * \mathcal{C}^{\geq 1}$,
   i.e., for any X, there is a triangle $U \longrightarrow X \longrightarrow V \longrightarrow U[1]$ with $U \in \mathcal{C}^{\leq 0}$ and $V \in \mathcal{C}^{\geq 1}$.

Heart of t-structure

Theorem (Beilinson-Bernstein-Deligne)

$(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$: a t-structure

1. We have an abelian category $\mathcal{H} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$ called the *heart*.
2. There exists a cohomological functor $H : \mathcal{C} \to \mathcal{H}$ with the universality: For any cohomological functor $F : \mathcal{C} \to \mathcal{A}$ such that $\mathcal{N} \subseteq \operatorname{Ker} F$, there uniquely exists an exact functor $F' : \mathcal{H} \to \mathcal{A}$ which makes the following diagram commutative.

$$\begin{array}{ccccc} \mathcal{N} & \longrightarrow & \mathcal{C} & \xrightarrow{H} & \mathcal{H} \\ & {\scriptstyle 0}\searrow & \downarrow{\scriptstyle F} & \swarrow{\scriptstyle F'} & \\ & & \mathcal{A} & & \end{array}$$

Here we set $\mathcal{N} := \operatorname{Ker} H$.

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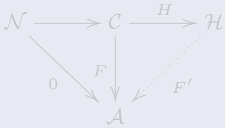

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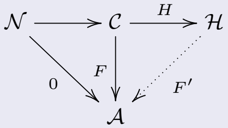

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Aim

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- $\mathcal{N}$: an extension-closed subcategory.

Aim

To construct an "exact" functor

$$Q : \mathcal{C} \to \mathcal{C}/\mathcal{N}$$

with a suitable universality which contains $\begin{cases} \text{Verdier quotient} \\ \text{heart of } t\text{-structure} \end{cases}$.

Verdier quotient vs Heart construction

(1) Verdier quotient $\mathcal{N} \to \mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{N}$

1. $\mathscr{S}_{\mathcal{N}} := \{f \mid Q(f) : \text{iso}\}$ is a multiplicative system (compatible with triangulation).
2. $\mathcal{N} = \operatorname{Ker} Q$ is a thick subcategory.
3. All appearing categories are triangulated.
4. All appearing functors are exact.

(2) Heart construction $\mathcal{N} \to \mathcal{C} \xrightarrow{H} \mathcal{H}$

1. $\mathscr{S}_{\mathcal{N}} := \{f \mid H(f) : \text{iso}\}$ is NOT a multiplicative system.
2. $\mathcal{N} = \operatorname{Ker} H$ is an extension-closed subcategory.
3. $\mathcal{C}$ is triangulated. $\mathcal{H}$ is abelian. What is $\mathcal{N}$?
4. The cohomological functor H is NOT "exact".

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The first problem

1. $\mathscr{S}_{\mathcal{N}} := \{f \mid Q(f) : \text{iso}\}$ is NOT a multiplicative system.

Lemma

- For the case of the Verdier quotient, the following equalities hold.

$$\begin{aligned}\mathscr{S}_{\mathcal{N}} &:= \{f \mid Q(f) : \text{iso}\} \\ &= \{f \mid \mathsf{Cone}(f) \in \mathcal{N}\} \\ &= \left\{ f \,\middle|\, \begin{array}{l} A \xrightarrow{x} B \xrightarrow{f} C \xrightarrow{y} A[1] \\ x, y \text{ factor through } \mathcal{N} \end{array} \right\}\end{aligned}$$

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Proposition (O)

For an extension-closed subcategory $\mathcal{N} \subseteq \mathcal{C}$,

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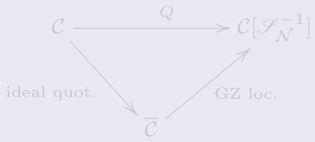
forms a multiplicative system $\overline{\mathscr{S}_{\mathcal{N}}}$ in $\overline{\mathcal{C}} := \mathcal{C}/[\mathcal{N}]$.
Thus, we have the Gabriel-Zisman localization



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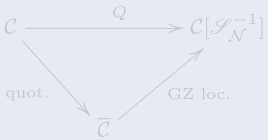

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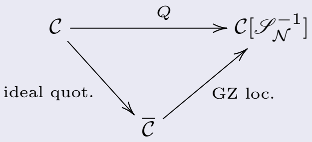

which is additive.

The third problem

3. $\mathcal{C}$ is triangulated. $\mathcal{H}$ is abelian. What is $\mathcal{N}$?

$\rightsquigarrow$ All appearing categories are extriangulated.

Definition (Nakaoka-Palu)

An *extriangulated category* is defined to be a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ of

- an additive category $\mathcal{C}$;
- an additive bifunctor $\mathbb{E} : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Ab}$, where Ab is the category of abelian groups;
- a correspondence $\mathfrak{s}$ which associates each equivalence class of a sequence $A \xrightarrow{x} B \xrightarrow{y} C$ in $\mathcal{C}$ to an element in $\mathbb{E}(C, A)$ for any $C, A \in \mathcal{C}$,

which satisfies some 'additivity' and 'compatibility'.
We call the sequence, the morphisms x and y an $\mathfrak{s}$-*conflation*, $\mathfrak{s}$-*inflation* and $\mathfrak{s}$-*deflation*.

The forth problem

4. The cohomological functor H is NOT "exact".

Theorem (Sakai)

For any cohomological functor $H : \mathcal{C} \to \mathcal{A}$, there exists a relative extriangulated structure $(\mathcal{C}, \mathbb{E}_H, \mathfrak{s}_H)$ on $\mathcal{C}$ such that

$$H : (\mathcal{C}, \mathbb{E}_H, \mathfrak{s}_H) \to \mathcal{A}$$

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can give rise to an exact functor.

The second problem

2. $\mathcal{N} = \operatorname{Ker} H$ is an extension-closed subcategory.

Theorem (O)

$\mathcal{N} = \operatorname{Ker} H$ is a thick subcategory in $(\mathcal{C}, \mathbb{E}_H, \mathfrak{s}_H)$.

To formulate our localization

$\mathcal{C}$: a triangulated category
$\mathcal{N}$: an extension-closed subcategory of $\mathcal{C}$

1. $\mathscr{S}_{\mathcal{N}} := \left\{ f \;\middle|\; \begin{array}{l} A \xrightarrow{x} B \xrightarrow{f} C \xrightarrow{y} A[1] \\ x, y \text{ factor through } \mathcal{N} \end{array} \right\}$ forms a multiplicative system in $\overline{\mathcal{C}}$.
2. A relative structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ on $\mathcal{C}$ naturally determined by $\mathcal{N}$.
3. The localization of the extriangulated category $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ by the thick subcategory $\mathcal{N}$.

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Relative structure determined by $\mathcal{N}$

Proposition (O)

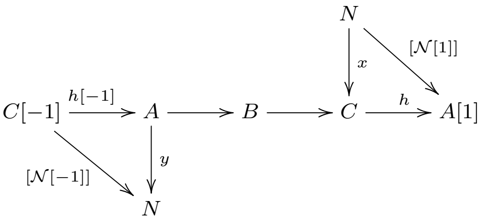
Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{C}$ an extension-closed subcategory. For any objects $A, C \in \mathcal{C}$, we define a subset of $\mathbb{E}(C, A)$ as follows.

- A subset $\mathbb{E}_{\mathcal{N}}(C, A)$ is defined as the set of morphisms $h : C \to A[1]$ satisfying the condition:

(Lex) For any morphism $N \xrightarrow{x} C$ with $N \in \mathcal{N}$, $h \circ x$ factors through an object in $\mathcal{N}[1]$.

(Rex) For any morphism $A \xrightarrow{y} N$ with $N \in \mathcal{N}$, $y \circ h[-1]$ factors through an object in $\mathcal{N}[-1]$.

We have an extriangulated structure $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ which are relative to $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$. Furthermore, $\mathcal{N}$ is thick in $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.



Localization of extriangulated categories

Theorem (Nakaoka-O-Sakai)

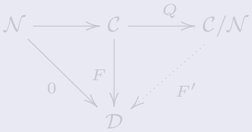
Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and $\mathcal{N} \subseteq \mathcal{C}$ a thick subcategory. Define the classes of morphisms:

- $\mathcal{L} := \{f \mid \mathsf{Cone}(f) \in \mathcal{N}\}$;
- $\mathcal{R} := \{f \mid \mathsf{CoCone}(f) \in \mathcal{N}\}$;
- $\mathscr{S}_{\mathcal{N}}$:= finite composition of morphisms belonging to $\mathcal{L} \cup \mathcal{R}$.

If the class $\mathscr{S}_{\mathcal{N}}$ satisfies some conditions $(*)$, we have an exact functor

$$(Q, \mu) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}})$$

with the universality: For any exact functor $(F, \phi) : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ such that $\mathcal{N} \subseteq \operatorname{Ker} F$, there uniquely exists an exact functor $(F', \phi') : (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}, \widetilde{\mathfrak{s}}) \to (\mathcal{D}, \mathbb{F}, \mathfrak{t})$ which makes the following diagram commutative.



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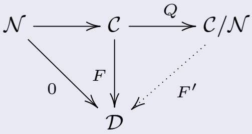

Our localization

Theorem (O)

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{C}$ an extension-closed subcategory.

(0) $\mathcal{N}$ is a thick subcategory in $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$.

(1) We have an extriangulated localization $(Q, \mu) : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$.

(2) $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ is a triangulated category.

(3) Suppose that $\mathcal{N}$ is functorially finite. Then, $\mathcal{N}$ satisfies $\mathcal{N} * \mathcal{N}[1] = \mathcal{C}$ in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ is an abelian category. Furthermore, the functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \widetilde{\mathcal{C}}_{\mathcal{N}}$ from the original triangulated category is cohomological.

$\mathcal{N}$	extension-closed	thick	$\mathcal{N} * \mathcal{N}[1] = \mathcal{C}$
$\mathcal{C}/\mathcal{N}$	extraingulated	triangulated	abelian

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(1) We have an extriangulated localization $(Q, \mu) : (\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}}) \to (\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$.

(2) $\mathcal{N}$ is thick in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $(\mathcal{C}/\mathcal{N}, \widetilde{\mathbb{E}}_{\mathcal{N}}, \widetilde{\mathfrak{s}}_{\mathcal{N}})$ is a triangulated category.

(3) Suppose that $\mathcal{N}$ is functorially finite. Then, $\mathcal{N}$ satisfies $\mathcal{N} * \mathcal{N}[1] = \mathcal{C}$ in the triangulated category $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ if and only if $(\mathcal{C}, \mathbb{E}_{\mathcal{N}}, \mathfrak{s}_{\mathcal{N}})$ is an abelian category. Furthermore, the functor $Q : (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \to \widetilde{\mathcal{C}}_{\mathcal{N}}$ from the original triangulated category is cohomological.

$\mathcal{N}$	extension-closed	thick	$\mathcal{N} * \mathcal{N}[1] = \mathcal{C}$
$\mathcal{C}/\mathcal{N}$	extraingulated	triangulated	abelian

Our localization

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Cohomological functor

Assume $\mathcal{N} * \mathcal{N}[1] = \mathcal{C}$.

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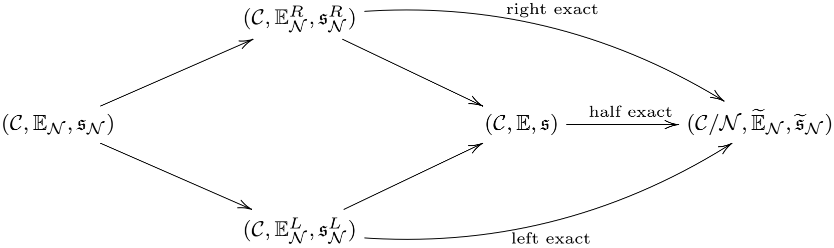
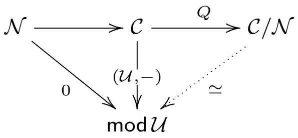


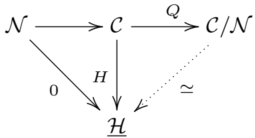
Examples

1. Verdier quotient.
2. The heart of a t-structure.
3. (Koenig-Zhu) The abelian quotient by a (2-)cluster tilting subcategory $\mathcal{N}$.
4. (Beligiannis, Buan-Marsh) Let $\mathcal{U}$ be a rigid contravariantly finite subcategory of $\mathcal{C}$ and consider the functor $(\mathcal{U}, -) : \mathcal{C} \to \mathsf{mod}\,\mathcal{U}$. Put $\mathcal{N} := \mathrm{Ker}(\mathcal{U}, -)$.

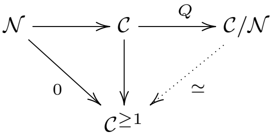


Examples

5. (Abe-Nakaoka) Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair of $\mathcal{C}$. Then, we have an abelian heart $\underline{\mathcal{H}}$ and a cohomological functor $H : \mathcal{C} \to \underline{\mathcal{H}}$. Put $\mathcal{N} := \operatorname{Ker} H$.



6. (Tattar) Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$ be a t-structure. Put $\mathcal{N} := \mathcal{C}^{\leq 0}$.



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