A new framework of partially additive algebraic geometry

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Outline

Introduction

$\mathbb{Z}$	$\mathbb{F}_q[X]$	(holomorphic functions on $\mathbb{C}$)
$\mathbb{Q}$	$\mathbb{F}_q(X)$	(meromorphic functions on $\mathbb{C}$)
?	curves over $\mathbb{F}_q$	curves over $\mathbb{C}$(Riemann Surface)
a prime $p \in \mathrm{Spec}(\mathbb{Z})$	$x \in \mathbb{F}_q$	$P \in \mathbb{C}$
$f(p) = [f \mod p] \in \mathbb{Z}/(p)$	$f(x) \in \mathbb{F}_q$	$f(P) \in \mathbb{C}$

(c.f. https://ncatlab.org/nlab/show/function+field+analogy)

Is there a field k such that $\mathbb{Z}$ is a ring of regular functions of a curve over k?

Partially additive ring A : a ring in which restricted pairs of elements are summable

$\to$ ideals, prime ideals, localization

$\to$ the Zariski topology on the set X of prime ideals of A, the sheaf of partial rings $\mathcal{O}_X$ on X,

$\to$ "affine partial scheme" $\operatorname{Spec} A = (X, \mathcal{O}_X)$

$\to$ "partial scheme"

$\to$ We construct

- The projective space $\mathbb{P}^n(A)$ as a partial scheme
- The general linear group $\mathbb{GL}_n(A)$ as a partial group partial scheme

Field of characteristic one

- (1951) Steinberg : An analogy between the characters of $\mathfrak{S}_n$ and those of $\mathrm{GL}_n(\mathbb{F}_q)$.
- (1957) J.Tits：*Les groupes de Chevalley sur le “corps de caractéristique 1”*. For any Chevalley group $G, G(\mathbb{F}_1)$ should be the Weyl group $W(G)$.
- (1992) Smirnov : An application of hypothetical $\mathbb{F}_1$-geometry to ABC-conjecture.
- (1995) Manin：based on the “beautiful recent idea of Deninger-Kurokawa”, he studies absolute motives, zeta functions over $\mathbb{F}_1$, and its application to Riemann conjecture.
- (2003) Kurokawa-Ochiai-Wakayama：“Absolute Derivations.”
- 黒川信重：「絶対数学」

- (1999) Soulé, (2010) Conne-Consani : a variety over $\mathbb{F}_1$ is (a functor $\mathrm{Ring} \to \mathrm{FinSet}$)+(an algebra $A_X/\mathbb{C}$)+(an affine scheme $X_{\mathbb{Z}}$)+(nat. trans. $X \to \mathrm{Hom}(A_X, - \otimes_{\mathbb{Z}} \mathbb{C})$)$+\cdots$ such that $\cdots$

- (1994) Kato, (2005) Deitmar : $\mathbb{F}_1$-schemes are defined as usual, but use commutative monoids instead of commutative rings,

- (2007) Durov : he uses a category of self-functors on $\mathcal{S}et$ (an algebraic monads) as the substitute for commutative rings

- (2007) Haran : using a category with some conditions (a generalization of the category of modules over a ring R), defines the set of prime ideals and Zariski topology on it.

- (2009) Borger : λ-structure on a scheme over $\mathbb{Z}$ is a descent data $\mathbb{Z} \to \mathbb{F}_1$,

- (2009) Toën-Vaquie : for any symmetric monoidal category $(\mathcal{C}, \otimes, 1)$, put $\mathrm{Aff}_{\mathcal{C}} = \mathcal{CM}on(\mathcal{C}^{op})$ and Grothendieck topology on it, results in $\mathbb{F}_1$-schemes when the category is $(\mathcal{S}et_0, \times, *)$

- (2012) Lorscheid : “Blueprints", (2013) Deitmar : “Sesquiads" : A blueprint is an equivalence relation on the semiring-semigroup $\mathbb{N}[A]$, where A is a commutative monoid. Then a scheme is defined analogously to the classical case.

$\mathbb{F}_1 = \{ 0, 1 \}$,

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}, \quad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

For them (who think of commutative monoids as “rings”),

$$(\text{An “}\mathbb{F}_1\text{-module”}) = (\text{based set})$$
$$(\text{A homomorphism between } \mathbb{F}_1\text{-modules}) = (\text{a base preserving map})$$

$$\mathbb{F}_1^n = \{ 0, e_1, \dots, e_n \} =: n_+ \text{ (a based set)}$$
$$\mathrm{End}(\mathbb{F}_1^n) = \{ \text{map between based sets } n_+ \to n_+ \}$$
$$\mathrm{GL}_n(\mathbb{F}_1) = \mathrm{Aut}(\mathbb{F}_1^n) = \{ \text{bijection between based sets } n_+ \to n_+ \} \cong \mathfrak{S}_n = W(\mathrm{GL}_n)$$

On the same footing,

$$\#\mathbb{P}^n(\mathbb{F}_q) = 1 + q + \cdots + q^n \longrightarrow n + 1 \ (q \to 1)$$

and

$$\#\mathbb{P}^n(\mathbb{F}_1) = n + 1$$

by counting the number of lines through origin in $\mathbb{F}_1^{n+1} = (n+1)_+$.

We give another explanation for these phenomena, which might be (hopefully) more reasonable.

module	$\otimes$	unit object	ring	geometry
$\mathcal{Ab}$	$\otimes$	$\mathbb{Z}$	commutative ring / $\mathcal{CRing}$	Varieties and Schemes — Grothendieck et. al.
$\mathcal{Set}$	$\times$	$* = \{1\}$	commutative monoid / $\mathcal{CMon}$	$\mathbb{F}_1$-Schemes — Deitmar (2005)
$\mathcal{PMon}$	$\otimes$	$\mathbb{F}_1^+$	"partial ring" / $\mathcal{PRing}$	"partial scheme"

Partial Monoids

A **partial monoid** is

$(A, 0)$: a based set , A_2: a subset of $A \times A$ and a map $+\colon A_2 \to A$

which enjoys the following properties

1. (Unit) $A \times \{0\} \cup \{0\} \times A \subseteq A_2$ and $a + 0 = a = 0 + a$,
2. (Commutative) $(a_1, a_2) \in A_2 \iff (a_2, a_1) \in A_2$ and $a_1 + a_2 = a_2 + a_1$,
3. (Associative)

$$(a_1, a_2) \in A_2, (a_1 + a_2, a_3) \in A_2 \iff (a_2, a_3) \in A_2, (a_1, a_2 + a_3) \in A_2$$

and $(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3)$.

If 3 is not required, A is called a **partial magma.**

A map $f\colon A \to B$ between partial monoids (magmas) is a **homomorphism** if

1. $f(0) = 0$,
2. $(a_1, a_2) \in A_2 \implies (f(a_1), f(a_2)) \in B_2$ and
3. $f(a_1 + a_2) = f(a_1) + f(a_2),\ \forall (a_1, a_2) \in A_2$

Let $\mathcal{PM}on$ ($\mathcal{PM}ag$) denote the category of partial monoids (magmas)

We have canonical fully faithful embeddings of categories

$$\begin{array}{ccccccc} \mathcal{S}et_0 & \longrightarrow & \mathcal{PM}on & \longleftarrow & \mathcal{A}b\mathcal{M}on & \longleftarrow & \mathcal{A}b\mathcal{G}rp \\ & & \big\downarrow & & & & \\ & & \mathcal{PM}ag & & & & \end{array}$$

Examples (Partial Monoids)

1. Every based set $(A, 0)$ is a partial monoid with $A_2 = A \times \{0\} \cup \{0\} \times A$.
2. Every commutative monoid $(A, 0)$ is a partial monoid with $A_2 = A \times A$. (So is an abelian group.)
3. Every partial monoid of order 2 is isomorphic to one of the following:

$$\mathbb{F}_1^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}, \quad \mathbb{F}_2^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}, \quad \mathbb{B}^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}.$$

(We have inclusions $\mathbb{F}_1^+ \hookrightarrow \mathbb{F}_2^+$ and $\mathbb{F}_1^+ \hookrightarrow \mathbb{B}^+$.)

4. For any abelian monoid M and any subset $A \subseteq M$ which contains 0, A is given a partial magma structure by putting

$$A_2 = \{ (a_1, a_2) \mid a_1 + a_2 \in A \} .$$

In general, A is not associative.
ex. $A = \{ 0, 1, 2, 3, 6 \} \subseteq \mathbb{N}$. Its "**associative closure**" is $A_a = \{ 0, 1, 2, 3, 4, 5, 6 \}$ since

$$\begin{aligned}(1+2)+3 &= 1+(2+3) \\ &= 2+(1+3)\end{aligned}$$

implies that $2 + 3 \in A_a$ and $1 + 3 \in A_a$.

Facts about $\mathcal{PMon}$ ($\mathcal{PMag}$)

- A homomorphism in $\mathcal{PMon}$ ($\mathcal{PMag}$) is monic if and only if it is an injective map (monic in $\mathcal{Set}$).
- There exist homomorphisms in $\mathcal{PMon}$ ($\mathcal{PMag}$) that are monic and epic but not surjective.So $\mathcal{PMon}$ ($\mathcal{PMag}$) is not a balanced category.
  ex. Two inclusions in
  $$\mathbb{F}_1^+ \longrightarrow \mathbb{N}^+ \longrightarrow \mathbb{Z}^+$$
  are both monic and epic.
- $\mathcal{PMon}$ ($\mathcal{PMag}$) has all small limits and colimits.
- $\mathcal{PMag}$ is a regular category, but $\mathcal{PMon}$ is not.

Partial Rings

A **partial ring** is a partial monoid equipped with a bilinear, commutative, associative product with identity.

addition + multiplication	$\mathcal{CMon}$	$\mathcal{PRing}$	$\mathcal{SRing}$	$\mathcal{CRing}$
addition	$\mathcal{Set}_0$	$\mathcal{PMon}$	$\mathcal{AbMon}$	$\mathcal{Ab}$

$$
\begin{array}{ccccccc}
\mathcal{CMon} & \longrightarrow & \mathcal{PRing} & \longleftarrow & \mathcal{SRing} & \longleftarrow & \mathcal{CRing} \\
| & & | & & | & & | \\
\mathcal{Set}_0 & \longrightarrow & \mathcal{PMon} & \longleftarrow & \mathcal{AbMon} & \longleftarrow & \mathcal{Ab}
\end{array}
$$

A **weak partial ring** is defined similarly, but starting with a partial magma.

Examples (Partial Rings)

1. A commutative monoid with absorbing element 0 is a partial ring.
2. A commutative semiring is a partial ring. (So is a commutative ring.)
3. A partial ring of order 2 is one of $\mathbb{F}_1, \mathbb{F}_2$ and $\mathbb{B}$.

$$\mathbb{F}_1^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & \end{array}, \quad \mathbb{F}_2^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}, \quad \mathbb{B}^+ = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \quad \mathbb{F}_1^\bullet, \mathbb{F}_2^\bullet, \mathbb{B}^\bullet = \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}.$$

4. According to an inaccurate investigation, there are 17 non-isomorphic partial rings of order 3. Among them, only one is a ring, five are semirings which are not rings, and the remaining 11 are partial rings that are not semirings.

5. The interval $[-1,1]$ is a weak partial subring of $\mathbb{R}$.
6. The interval $[0,1]$ is a partial subring of $\mathbb{R}$.
7. The unit disk $D = \{ z \mid |z| \leq 1 \}$ is a weak partial subring of $\mathbb{C}$. (The inclusion $D \hookrightarrow \mathbb{C}$ is the canonical morphism of the associative closure.)
8. Partial rings sitting between $\langle x_1, \ldots, x_n \rangle$ and $\mathbb{N}[x_1, \ldots, x_n]$ are quite useful, where $\langle x_1, \ldots, x_n \rangle$ is the commutative monoid generated by indeterminates $x_1, \ldots, x_n$ with an absorbing element 0 adjoined, and $\mathbb{N}[x_1, \ldots, x_n]$ denotes the polynomial semiring of n indeterminates.
   For example if $Z = \mathbb{F}_1\langle x, y \mid x + y \rangle$ denotes the smallest partial subring of $\mathbb{N}[x, y]$ which contains $\langle x, y \rangle$ and in which $x + y$ exists,then there is a bijection

$$\mathrm{Hom}_{\mathcal{PR}ing}(Z, A) \cong A_2$$

   for any partial ring A.

A-modules

Let A be a partial ring. An A**-module** is a partial monoid M which admits a bilinear, associative and unital action of A on it. More precisely, a partial monoid M is an A-module if $a \cdot m \in M$ is determined for each $a \in A$ and $m \in M$, and

1. If $a \in A$ then $a \cdot 0 = 0$.
2. If $a \in A$ and $(m_1, m_2) \in M_2$, then $(a \cdot m_1, a \cdot m_2) \in M_2$ and $a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$.
3. If $m \in M$ then $0 \cdot m = 0$.
4. If $(a_1, a_2) \in A$ and $m \in M$, then $(a_1 \cdot m, a_2 \cdot m) \in M_2$ and $(a_1 + a_2) \cdot m = a_1 \cdot m + a_2 \cdot m$.
5. If $a, b \in A$ and $m \in M$, then $(ab) \cdot m = a \cdot (b \cdot m)$.
6. If $m \in M$ then $1 \cdot m = m$.

Examples (A-modules)

1. Direct product

$$A^n = \{\, (a_1, \dots, a_n) \mid a_i \in A,\ \forall i \,\},$$
$$(A^n)_2 = \{\, (a_1, \dots, a_n; b_1, \dots, b_n) \in (A^n)^2 \mid (a_i, b_i) \in A_2,\ \forall i \,\}.$$

2. Summable n-tuples

$$A_n = \{\, (a_1, \dots, a_n) \in A^n \mid a_1 + \cdots + a_n \text{ is defined in } A \,\},$$
$$(A_n)_2 = A_{2n}.$$

This is a nice approximation to the dual of A^n.

Localization

Let A be a partial ring and S be a multiplicative subset. Let M be an A-module. We put

$$S^{-1}M = \left\{ \frac{m}{s} \,\middle|\, m \in M, s \in S \right\}$$
$$(S^{-1}M)_2 = \left\{ \left(\frac{m}{s}, \frac{n}{t}\right) \in S^{-1}M^2 \,\middle|\, \exists u \in S \text{ s.t. } (utm, usn) \in M_2 \right\}.$$

Then $S^{-1}M$ has the universality as usual.

Partial Schemes

An A-submodule of a partial ring A is called an **ideal.** An ideal $P \subseteq A$ is called a **prime ideal** if $A \setminus P$ is multiplicatively closed. Let $X = X_A$ denotes the set of prime ideals of A. We endow X a topology generated by

$$D(f) = \{ P \in X \mid f \notin P \}, f \in A.$$

For each open set $U \subseteq X$, we put $\mathcal{O}'(U) = S_U^{-1} A$ where $S_U = \{ s \in A \mid s \notin P, \forall P \in U \}$. Then $\mathcal{O}'_X$ is a presheaf over the space X. Its sheafification is denoted by $\mathcal{O}_X$. Now the partial-ringed space $(X, \mathcal{O}_X)$ is called an **affine partial scheme** and denoted by $\operatorname{Spec} A$. Finally, a **partial scheme** is a locally partial-ringed space that is locally isomorphic to an affine partial scheme.

1. X_A is quasi-compact for any partial ring A.
2. For any $f \in A$, there is a natural homomorphism $\mathcal{O}_X(D(f)) \to S_f^{-1}A$, which is injective.
3. There is a bijection

$$\mathrm{Hom}_{\mathcal{P}ar\mathcal{S}ch}(\mathrm{Spec}\, A, \mathrm{Spec}\, B) \cong \mathrm{Hom}_{\mathcal{PR}ing}(B, \Gamma(\mathcal{O}_{X_A}(\mathrm{Spec}\, A)).$$

4. If F is a "partial field", then $\Gamma(\mathcal{O}_{X_F}(\mathrm{Spec}\, F))$ is naturally isomorphic to F, so that

$$\mathrm{Hom}_{\mathcal{P}ar\mathcal{S}ch}(\mathrm{Spec}\, F, \mathrm{Spec}\, B) \cong \mathrm{Hom}_{\mathcal{PR}ing}(B, F).$$

$\mathbb{P}^n$

Let B be a partial subring of $\mathbb{N}[y_0, \ldots, y_n]$ that contains $\langle y_0, \ldots, y_n \rangle$;

$$\langle y_0, \ldots, y_n \rangle \subseteq B \subseteq \mathbb{N}[y_0, \ldots, y_n].$$

We give a grading to B so that $\deg y_i = 1$. We put $B^{(i)} = S_i^{-1}B$, where $S_i = \{ y_i^r \mid r \in \mathbb{N} \}$and let $A^{(i)}$ denote the 0-th part of $B^{(i)}$.If $T_j = \{ (y_j/y_i)^r \mid r \in \mathbb{N} \}$ denotes the multiplicative subset of $A^{(i)}$, we have $T_i^{-1}A^{(j)} = T_j^{-1}A^{(i)} =: A^{(i,j)}$. Now put $X_i = \operatorname{Spec} A^{(i)}$ and glue them together on open partial subschemes $\operatorname{Spec}(T_j^{-1}A^{(i)}) \subseteq X_i$. The resulting partial scheme is denoted by $\mathbb{P}^n_B$.

Remark. The above construction is a direct translation of the corresponding part of Mumford's Red Book.

Now let $B = \mathbb{F}_1\langle y_0, \dots, y_n \mid y_0 + \cdots + y_n\rangle$. (This means that we are using A_{n+1} as the "ambient space".) Then

$$A^{(i)} = \{\, \text{subsum of } (y_0/y_i + \cdots + 1 + \cdots + y_n/y_i)^r \mid r \in \mathbb{N} \,\}$$

and

$$A^{(i,j)} = \{\, (\text{subsum of } (y_0/y_i + \cdots + 1 + \cdots + y_n/y_i)^r) \times (y_i/y_j)^s \mid r, s \in \mathbb{N} \,\}.$$

If F is a partial field, then there is a one-to-one correspondence

$$\left\{ \begin{array}{c} F\text{-valued points} \\ \operatorname{Spec} F \to X_i \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{partial ring homomorphisms} \\ A^{(i)} \to F \end{array} \right\}$$

We put $A^{(i)}(F) = \operatorname{Hom}_{\mathcal{PRing}}(A^{(i)}, F)$.

Then we have

$$\left\{ \begin{array}{c} F\text{-valued points} \\ \operatorname{Spec} F \to \mathbb{P}^n \end{array} \right\} \longleftrightarrow \coprod_{i=1}^{n} A_i(F)/\sim$$

as sets, where for $v_i \in A_i(F)$ and $v_j \in A_j(F)$, $v_i \sim v_j$ if there exists a homomorphism $v\colon A^{(i,j)} \to F$ such that $v|_{A^{(i)}} = v_i$ and $v|_{A^{(j)}} = v_j$. So we may write

$$\mathbb{P}^n(F) = \coprod_{i=1}^{n} A_i(F)/\sim .$$

For any subset $\{ i_1, \ldots, i_r \} \subseteq \{ 0, 1, \ldots, n \}$, we put

$$A_{i_1,\ldots,i_r}(F) = \mathrm{Hom}_{\mathcal{PR}ing}\left(A^{(i_1,\ldots,i_r)}, F\right).$$

Then as sets,

$$A_{i_1,\ldots,i_r}(F) \cong \left(\begin{array}{l} n\text{-tuples } (x_1, \ldots, x_n) \text{ for which} \\ 1 + x_1 + \cdots + x_n \text{ can be calculated in } F \\ \text{and } x_1, \ldots, x_{r-1} \neq 0 \end{array} \right).$$

For $F = \mathbb{F}_q$ (including the case $q = 1$!), we have that

$$\# A_{i_1,\ldots,i_r}(F) = (\kappa - 1)^{r-1} \kappa^{n-r+1},$$

where $\kappa = \kappa(F)$ denotes the number of elements of F which is summable with 1.

Now we can calculate $\#\mathbb{P}^n(F)$ as

$$\begin{aligned}
\#\mathbb{P}^n(F) &= \sum_{r=1}^{n+1}(-1)^{r-1}\binom{n+1}{r}(\kappa-1)^{r-1}\kappa^{n-r+1} \\
&= \frac{1}{\kappa-1}\left(\sum_{r=0}^{n+1}(-1)^{r-1}\binom{n+1}{r}(\kappa-1)^{r-1}\kappa^{n-r+1}+\kappa^{n+1}\right) \\
&= -\frac{(\kappa-(\kappa-1))^{n+1}-\kappa^{n+1}}{\kappa-1} \\
&= \frac{\kappa^{n+1}-1}{\kappa-1} = \kappa^n+\cdots+\kappa+1.
\end{aligned}$$

Of course, $\kappa(\mathbb{F}_q) = q$, where $\mathbb{F}_q$ denotes the finite field with q elements, and $\kappa(\mathbb{F}_1) = 1$.

$\mathbb{GL}_n$

Let A be a partial ring. An A-submodule of A_n given by

$$A_{(n)} = \left\{ (a_1, \ldots, a_n) \in A^n \;\middle|\; \begin{array}{l} (c_1 a_1, \ldots, c_n a_n) \in A_n, \\ \forall (c_1, \ldots, c_n) \in A^n \end{array} \right\}$$

is isomorphic to the dual A-module of A^n.

Accordingly, the A-module of the A-homomorphisms $A^n \to A^m$ is isomorphic to the A-module $M_{m,n}(A)$ of $m \times n$-matrices whose rows are in $A_{(n)}$.

Unfortunately, the correspondence

$$A \mapsto A_{(n)}$$

is not a ($\mathcal{S}et$-valued) functor.

Nor is the correspondence

$$A \mapsto M_{m,n}(A).$$

We say that a partial ring A is **good** if $A_n = A_{(n)}$.

1. Abelian monoids and commutative rings are good.
2. Every good partial field is a semifield or a commutative group.
3. There are good partial rings that are not abelian monoids nor commutative rings — $[0, 1]$ and $\mathbb{N}_1[x]$ are such.

The correspondence

$$A \mapsto A_n$$

is a ($\mathcal{S}et$-valued) functor. So is the correspondence

$$A \mapsto M'_{m,n}(A),$$

where $M'_{m,n}(A)$ is the A-module of $m \times n$ matrices whose rows are in A_n.

We have that

1. $M_n(A) = M_{n,n}(A)$ is a non-commutative monoid.
2. $M'_n(A) = M'_{n,n}(A)$ is a **non-commutative partial magma.**
3. $M_n(A) = M'_n(A)$ for a good partial ring A.

Let $GL_n(A)$ denote the subset of $M_n(A)$ which consists of invertible matrices. Put $GL'_n(A) = GL_n(A) \cap M'_n(A)$.

Then we have that

1. $GL_n(A)$ is a non-commutative group.
2. $GL'_n(A)$ is a **non-commutative partial group.**
3. $GL_n(A) = GL'_n(A)$ for a good partial ring A.

Theorem

There exists a representable functor $\mathbb{GL}_n \colon \mathcal{PR}ing \to \mathcal{PG}rp$ which enjoys the following properties:

1. *its restriction to the category of good partial rings factors through $\mathbb{GL}_n \colon \mathcal{PR}ing \to \mathcal{G}rp$.*
2. *$\mathbb{GL}_n(A)$ is the group of n-th general linear group with entries in A, if A is a commutative rings with 1, and*
3. *$\mathbb{GL}_n(\mathbb{F}_1) = \mathfrak{S}_n$ is n-th symmetric group.*

Thank you for your attention!