On IE-closed subcategories

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The 54th Symposium on Ring Theory and Representation Theory,
6 September, 2022

This is a joint work with Haruhisa Enomoto (Osaka Metropolitan University).

Convention

- k : a field.
- Λ : a finite dimensional k-algebra.
- $\operatorname{mod}\Lambda$: the category of finitely generated right Λ-module.
- All modules are finitely generated.
- All subcategories are full, additive and closed under summands.

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In representation theory of finite dimensional algebra, it is fundamental to study subcategories of mod Λ.

Today's topic

- Functorial finiteness
- Classification

of IE-closed subcategories

Definition

Let $\mathcal{C}$ be a subcategory of mod Λ.

1. $\mathcal{C}$ is a torsion class if $\mathcal{C}$ is closed under taking extensions and quotients.
2. $\mathcal{C}$ is a torsion-free class if $\mathcal{C}$ is closed under taking extensions and submodules.
3. $\mathcal{C}$ is a wide subcategory if $\mathcal{C}$ is closed under taking extensions, kernels and cokernels.

Definition

A subcategory $\mathcal{C}$ of $\operatorname{mod} \Lambda$ is an IE-closed subcategory if $\mathcal{C}$ is closed under taking extensions and images, that is, $\operatorname{Im} f \in \mathcal{C}$ for any morphism f in $\mathcal{C}$.

- Torsion classes, torsion-free classes and wide subcategories are IE-closed subcategories.
- Subcategories closed under images are considered by Auslander and Smalø in [AS1].

Proposition

TFAE for a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$.

1. $\mathcal{C}$ *is an IE-closed subcategory of* $\operatorname{mod}\Lambda$.
2. *There exist a torsion class* $\mathcal{T}$ *and a torsion-free class* $\mathcal{F}$ *in* $\operatorname{mod}\Lambda$ *such that* $\mathcal{C} = \mathcal{T} \cap \mathcal{F}$.

In this case, $\mathcal{C} = \mathsf{T}(\mathcal{C}) \cap \mathsf{F}(\mathcal{C})$ *holds.*

We denote by $\mathsf{T}(\mathcal{C})$ (resp. $\mathsf{F}(\mathcal{C})$) the smallest torsion class (resp. torsion-free class) containing $\mathcal{C}$.

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The notion of functorial finiteness appears from the perspectives of

- functor categories [Aus],
- Auslander-Reiten sequences [AS2],

and so on.

Example

TFAE for a torsion class $\mathcal{T} \subseteq \operatorname{mod} \Lambda$.

- $\mathcal{T}$ is functorially finite in $\operatorname{mod} \Lambda$.
- There exists $M \in \mathcal{T}$ s.t. $\mathcal{T} = \operatorname{Fac} M$.
- $\mathcal{T}$ has enough projectives as an exact category.

Example

TFAE for a wide subcategory $\mathcal{W} \subseteq \operatorname{mod} \Lambda$.

- $\mathcal{W}$ is functorially finite in $\operatorname{mod} \Lambda$.
- There exists a f.d. k-algebra Γ s.t. $\mathcal{W} \simeq \operatorname{mod} \Gamma$.
- $\mathcal{W}$ has a progenerator as an exact category.

Definition

A f.d.algebra Λ is τ-tilting finite if the set of functorially finite torsion classes in $\operatorname{mod}\Lambda$ is a finite set.

Theorem (Demonet-Iyama-Jasso)

TFAE for a f.d.algebra Λ.

1. Λ *is* τ*-tilting finite.*
2. *The set of torsion classes in* $\operatorname{mod}\Lambda$ *is a finite set.*
3. *Every torsion class in* $\operatorname{mod}\Lambda$ *is functorially finite.*

Theorem (Enomoto-S)

TFAE for a f.d.algebra Λ.

1. Λ *is* τ*-tilting finite.*
2. *The set of IE-closed subcategories of* $\operatorname{mod}\Lambda$ *is a finite set.*
3. *Every IE-closed subcategory of* $\operatorname{mod}\Lambda$ *is functorially finite.*

Theorem (Auslander-Reiten)

TFAE for a f.d.algebra Λ.

1. Λ *is of finite representation type.*
2. *The set of additive subcategories of* $\operatorname{mod}\Lambda$ *closed under summands is a finite set.*
3. *Every additive subcategory of* $\operatorname{mod}\Lambda$ *closed under summands is functorially finite.*

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Theorem (Adachi-Iyama-Reiten)

There exist bijective correspondences between:

1. *The set of functorially finite torsion classes in* mod Λ,
2. *The set of iso-classes of basic support* τ*-tilting modules.*

A functorially finite torsion class $\mathcal{T}$ recovers form its Ext-progenerator $\mathbf{P}(\mathcal{T})$:

$$\mathcal{T} = \mathsf{Fac}(\mathbf{P}(\mathcal{T}))$$

In the rest, we assume that Λ is hereditary.
A functorially finite IE-closed subcategory $\mathcal{C}$ recovers form its Ext-progenerator $\mathbf{P}(\mathcal{C})$ and Ext-injective cogenerator $\mathbf{I}(\mathcal{C})$:

$$\mathcal{C} = \mathsf{Fac}(\mathbf{P}(\mathcal{C})) \cap \mathsf{Sub}(\mathbf{I}(\mathcal{C}))$$

Theorem (Enomoto-S)

There exist bijective correspondences between:

1. *The set of functorially finite IE-closed subcategories of* $\operatorname{mod} \Lambda$,
2. *The set of isomorphism classes of basic twin rigid modules.*

Definition

A pair (P, I) of Λ-modules is a twin rigid module if it satisfies

- P and I are rigid, that is, $\operatorname{Ext}^1_\Lambda(P, P) = 0$ and $\operatorname{Ext}^1_\Lambda(I, I) = 0$.
- There are short exact sequences

$$0 \to P \to I^0 \to I^1 \to 0$$
$$0 \to P_1 \to P_0 \to I \to 0$$

with $P_0, P_1 \in \operatorname{add} P$ and $I^0, I^1 \in \operatorname{add} I$.

Fix a rigid Λ-module P and set $\Gamma_P = \operatorname{End}_\Lambda(P)$.
There is a functor $\operatorname{Hom}_\Lambda(P, -) \colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma_P$.

Proposition (Enomoto-S)

Let (P, I) be a twin rigid module. Then

1. $\operatorname{Hom}_\Lambda(P, I)$ *is a tilting Γ_P-module.*
2. *The equality $|P| = |I|$ holds.*

Proposition (Enomoto-S)

The functor $\mathrm{Hom}_\Lambda(P, -)$ *induces a bijective correspondence between*

1. *The set of isomorphism classes of twin rigid modules* (P, I)*,*
2. *The set of isomorphism classes of tilting* Γ_P*-modules contained in* $\mathsf{Sub}(DP)$*.*

- Taking advantage of this, we introduce completion and mutation of twin rigid modules as an analogue of Bongartz completion and tilting mutation in classical tilting theory.
- If Λ is of finite representation type, we can calculate all twin rigid modules by mutation.

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We give examples of IE-closed subcategories by using the following tools:

- String Applet: `https://www.math.uni-bielefeld.de/~jgeuenich/string-applet/`
- AR quiver calculator: `https://haruhisa-enomoto.github.io/codes/`

Thank you for listening.

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