Characterization of eventually periodic modules and its applications

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1. Motivation

- R : a left noetherian ring
- Modules = finitely generated left modules

Definition (Buchweitz '86, Orlov '04)

$\mathcal{D}_{\mathrm{sg}}(R) := \mathcal{D}^{\mathrm{b}}(R\text{-mod})/\mathcal{K}^{\mathrm{b}}(R\text{-proj})$: the *singularity category* of R

- $\mathrm{pd}_R M < \infty \iff M \cong 0$ in $\mathcal{D}_{\mathrm{sg}}(R)$

Question

Can we use $\mathcal{D}_{\mathrm{sg}}(R)$ to describe homological properties of M with $\mathrm{pd}_R M = \infty$?

- In the rest, R : a left artin ring

Definition

(1) ${}_RM$: *periodic* $\overset{\text{def}}{\iff}$ $\Omega_R^p(M) \cong M$ in $R\text{-mod}$ $(\exists p > 0)$

The least such p is called the *period* of M

(2) ${}_RM$: *eventually periodic* $\overset{\text{def}}{\iff}$ $\Omega_R^n(M)$ is non-zero periodic $(\exists n \geq 0)$

<u>Remark</u> Eisenbud ('80) dealt with eventually periodic modules

Example

$$\Lambda : \ 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \quad \alpha\beta\alpha = 0; \qquad P_1 = \begin{smallmatrix}1\\2\\1\end{smallmatrix} \qquad P_2 = \begin{smallmatrix}2\\1\\2\\1\end{smallmatrix}$$

- $S_1 = 1$ is eventually periodic, because

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \begin{smallmatrix}2\\1\\2\\1\end{smallmatrix} & & \longrightarrow & & \begin{smallmatrix}2\\1\\2\\1\end{smallmatrix} & & \longrightarrow & & \begin{smallmatrix}1\\2\\1\end{smallmatrix} & \longrightarrow 1 \longrightarrow 0 \\
& & \searrow & & \nearrow & & \searrow & & \nearrow & & \\
& & & \begin{smallmatrix}2\\1\end{smallmatrix} & & & & \begin{smallmatrix}2\\1\end{smallmatrix} & & &
\end{array}$$

Definition (B '86)

$$\widehat{\mathrm{Ext}}_R^i(M,M) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(R)}(M, M[i]) \quad (i \in \mathbb{Z})$$

is called the *Tate cohomology group* of ${}_R M$

<u>Remark</u> He observed that $\widehat{\mathrm{Ext}}_{\mathbb{Z}G}^i(\mathbb{Z},\mathbb{Z}) \cong \widehat{\mathrm{H}}^i(G,\mathbb{Z})$ for a finite group G

- $\widehat{\mathrm{Ext}}_R^{\bullet}(M,M) = \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{Ext}}_R^i(M,M)$: the *Tate cohomology ring* of ${}_R M$

<u>Remark</u> $\mathrm{pd}_R M < \infty \iff \widehat{\mathrm{Ext}}_R^{\bullet}(M,M) = 0$

Theorem (U '21)

Λ : a fin. dim. Gorenstein algebra (i.e. $\mathrm{id}_\Lambda \Lambda$, $\mathrm{id}\, \Lambda_\Lambda < \infty$). TFAE for ${}_\Lambda M$.

1. M is eventually periodic
2. $\widehat{\mathrm{Ext}}_\Lambda^{\bullet}(M, M)$ has a non-zero homogeneous invertible element of positive degree

Our aim is to

extend the above theorem to more general rings!

Remark Eventually periodic modules have been characterized by Croll ('13) and Bergh ('06)

2. Main result

Theorem (U '22)

R : a left artin ring, and $p > 0$. TFAE for ${}_RM$.

1. $\Omega_R^{n+p}(M) \cong \Omega_R^n(M) (\neq 0)$ in $R\text{-mod}$ for some $n \geq 0$
2. $\widehat{\mathrm{Ext}}_R^{\bullet}(M, M)$ has a homog. invertible element $\chi (\neq 0)$ with $\deg \chi = p$

Remark $\widehat{\mathrm{Ext}}_R^{\bullet}(M, M)$ is related only to p in $\Omega_R^{n+p}(M) \cong \Omega_R^n(M)$

Proof The existence of an invertible element χ with $\deg \chi = p$

$\iff$ $M \cong M[p]$ in $\mathcal{D}_{\mathrm{sg}}(R)$

$\iff$ $\Omega_R^{l+p}(M) \cong \Omega_R^l(M)$ in $R\text{-}\underline{\mathrm{mod}}$ $(\exists l \geq 0)$

$\iff$ $\Omega_R^{n+p}(M) \cong \Omega_R^n(M)$ in $R\text{-mod}$ $(\exists n \geq 0)$

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(R)}(M, M[p]) \cong \lim_{l \to \infty} \underline{\mathrm{Hom}}_R(\Omega_R^{l+p}(M), \Omega_R^l(M))$$ (Keller-Vossieck '87)

Definition (Küpper '10)

Λ : *eventually periodic* $\overset{\text{def}}{\Longleftrightarrow}$ ${}_{\Lambda^{\mathrm{e}}}\Lambda$ is eventually periodic, where $\Lambda^{\mathrm{e}} = \Lambda \otimes \Lambda^{\mathrm{op}}$

<u>Recall</u> Λ : *periodic* $\overset{\text{def}}{\Longleftrightarrow}$ ${}_{\Lambda^{\mathrm{e}}}\Lambda$ is periodic

Example

$$\Lambda : \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \quad \alpha\beta\alpha = 0 \quad \text{and} \quad \Gamma : \; \alpha \circlearrowright 1 \xrightarrow{\beta} 2 \quad \alpha^2 = 0$$

- $\Omega^2_{\Lambda^{\mathrm{e}}}(\Lambda)$ and $\Omega^2_{\Gamma^{\mathrm{e}}}(\Gamma)$ are the first periodic syzygies
- Γ is Gorenstein, while Λ is not Gorenstein

Definition (Wang '15)

$\widehat{\mathrm{HH}}^i(\Lambda) := \widehat{\mathrm{Ext}}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$: the *Tate-Hochschild cohomology group* of Λ

Theorem (W '21)

$\widehat{\mathrm{HH}}^{\bullet}(\Lambda) = \widehat{\mathrm{Ext}}^{\bullet}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$ is graded commutative

- Applying our main result to ${}_R M = {}_{\Lambda^{\mathrm{e}}}\Lambda$, we have

Corollary

TFAE for a finite dimensional algebra Λ

1. Λ is eventually periodic
2. $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ has a homog. invertible element $\chi \neq 0$ with $\deg \chi > 0$

In this case, $\widehat{\mathrm{HH}}^{\bullet}(\Lambda) \cong \widehat{\mathrm{HH}}^{\geq 0}(\Lambda)[\chi^{-1}]$

3. Applications

- There are two results obtained from the corollary
- Let Λ and Γ be finite dimensional algebras over a field k

Definition (W '15)

$({}_\Lambda M_\Gamma, {}_\Gamma N_\Lambda)$ *defines a singular equivalence of Morita type with level* (*SEML*) $l \geq 0$
$\overset{\text{def}}{\iff}$

1. ${}_\Lambda M$, M_Γ, ${}_\Gamma N$, N_Λ are projective
2. $M \otimes_\Gamma N \cong \Omega^l_{\Lambda^e}(\Lambda)$ in Λ^e-$\underline{\mathrm{mod}}$ and $N \otimes_\Lambda M \cong \Omega^l_{\Gamma^e}(\Gamma)$ in Γ^e-$\underline{\mathrm{mod}}$

Theorem (W '15)

1. $\Lambda \overset{\text{SEML}}{\sim} \Gamma \implies \mathcal{D}_{\mathrm{sg}}(\Lambda) \cong \mathcal{D}_{\mathrm{sg}}(\Gamma)$ and $\widehat{\mathrm{HH}}^\bullet(\Lambda) \cong \widehat{\mathrm{HH}}^\bullet(\Gamma)$
2. $\Lambda \overset{\text{der}}{\sim} \Gamma \implies \Lambda \overset{\text{SEML}}{\sim} \Gamma$

• There are many invariants under SEML such as $\mathrm{fin.dim} < \infty$ (W '15), finiteness condition (Skartsæterhagen '16), being syzygy-finite, being Igusa-Todorov, injectives generation, and projectives cogenerarion (Qin '21)

Theorem (Erdmann-Skowroński '08)

Assume $\Lambda \overset{\mathrm{der}}{\sim} \Gamma$. If Λ is connected and periodic, then so is Γ. Moreover, their periods coincide.

Theorem (U '22)

Assume $\Lambda \overset{\mathrm{SEML}}{\sim} \Gamma$. If Λ is eventually periodic, then so is Γ. Moreover, the periods of their periodic syzygies coincide.

• The key is the fact that $\widehat{\mathrm{HH}}^{\bullet}(\Lambda) \cong \widehat{\mathrm{HH}}^{\bullet}(\Gamma)$

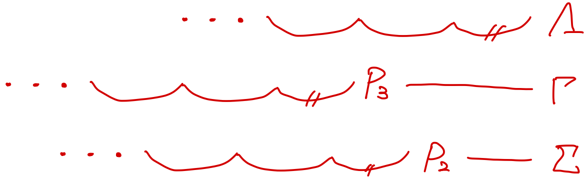
Example

$$\Lambda: \; \alpha \circlearrowright 1 \qquad \alpha^2 = 0,$$

$$\Gamma: \; \alpha \circlearrowright 1 \xrightarrow{\beta} 2 \qquad \alpha^2 = 0 = \beta\alpha,$$

$$\Sigma: \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad \alpha\beta\alpha = 0$$

- $\Lambda \overset{\text{SEML}}{\sim} \Gamma \overset{\text{SEML}}{\sim} \Sigma$, and Λ is periodic $\implies$ Γ and Σ are eventually periodic
- Λ, $\Omega^3_{\Gamma^e}(\Gamma)$, $\Omega^2_{\Sigma^e}(\Sigma)$ are the first periodic syzygies



- Assume now that $k = \overline{k}$ $\quad (\Longrightarrow \mathrm{pd}_{\Lambda^{\mathrm{e}}}\Lambda = \mathrm{gl.dim}\Lambda)$
- $\mathcal{CN} :=$ { finite dimensional connected Nakayama algebras }
- $\Lambda \in \mathcal{CN}$ is Morita equivalent to kQ/I, where Q is one of

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow e \qquad \text{and} \qquad Z_e : \; e \nearrow 1 \longrightarrow 2 \searrow 3 \swarrow \cdots \nwarrow e$$

- Thanks to Asashiba ('99), Shen ('14,'15) and Qin ('21), we obtain

Proposition

$\{\ k,\ kZ_e/R^N \mid e \geq 1, N \geq 2\ \}$ is a complete set of representatives of equivalence classes in $\mathcal{CN}/ \overset{\text{SEML}}{\sim}$

Corollary

$\Lambda \in \mathcal{CN}$ is eventually periodic $\iff$ $\operatorname{gl.dim} \Lambda = \infty$

Proof It suffices to show ($\Leftarrow$).

$$\operatorname{gl.dim} \Lambda = \infty \implies \Lambda \overset{\text{SEML}}{\sim} kZ_e/R^N, \text{ and } kZ_e/R^N \text{ is periodic}$$

$$\implies \Lambda \text{ is eventually periodic}$$