A BIJECTION BETWEEN SILTING SUBCATEGORIES AND BOUNDED HEREDITARY COTORSION PAIRS

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Abstract. In a triangulated category, there exists a bijection between silting subcategories and bounded co-t-structures. In this article, as a generalization of this result, we give a bijection between silting subcategories and bounded hereditary cotorsion pairs in an extriangulated category. Moreover, we prove that our result recovers a bijection between basic tilting modules and contravariantly finite resolving subcategories for a finite dimensional algebra with finite global dimension.

Throughout this article, we assume that every category is skeletally small, that is, the isomorphism classes of objects form a set. In addition, all subcategories are assumed to be full and closed under isomorphisms.

The notion of silting subcategories was firstly introduced by Keller and Vossieck [5].

Definition 1. Let $\mathcal{D}$ be a triangulated category with shift functor Σ. A subcategory $\mathcal{M}$ of $\mathcal{D}$ is called a *silting subcategory* if it satisfies the following conditions.

- $\mathcal{M}$ is closed under direct summands.
- $\mathcal{D}(\mathcal{M}, \Sigma^k\mathcal{M}) = 0$ for each $k \geq 1$.
- $\mathcal{D} = \mathsf{thick}\mathcal{M}$, where $\mathsf{thick}\mathcal{M}$ is the smallest thick subcategory containing $\mathcal{M}$.

Bondarko ([3]) and Pauksztello ([8]) independently introduced co-t-structures as an analog of t-structures.

Definition 2. Let $\mathcal{D}$ be a triangulated category with shift functor Σ. A pair $(\mathcal{U}, \mathcal{V})$ of subcategories of $\mathcal{D}$ is called a *co-t-structure* on $\mathcal{D}$ if it satisfies the following conditions.

- $\mathcal{U}$ and $\mathcal{V}$ are closed under direct summands.
- For each $D \in \mathcal{D}$, there exists a triangle $\Sigma^{-1}U \to D \to V \to U$ such that $U \in \mathcal{U}$ and $V \in \mathcal{V}$.
- $\mathcal{D}(\Sigma^{-1}\mathcal{U}, \mathcal{V}) = 0$.
- $\mathcal{U}$ is closed under a negative shift, that is, $\Sigma^{-1}\mathcal{U} \subseteq \mathcal{U}$.

A co-t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathcal{D}$ is said to be *bounded* if $\cup_{n\in\mathbb{Z}}\Sigma^n\mathcal{U} = \mathcal{D} = \cup_{n\in\mathbb{Z}}\Sigma^n\mathcal{V}$.

Bondarko ([3]) and Mendoza–Santiago–Sáenz–Souto ([6]) gave the following result.

Theorem 3 ([3, 6])**.** *Let $\mathcal{D}$ be a triangulated category. Then there exist mutually inverse bijections between the set of silting subcategories of $\mathcal{D}$ and the set of bounded co-t-structures on $\mathcal{D}$.*

The detailed version of this article has been published in [1].

The aim of this article is to generalize Theorem 3 to extriangulated categories introduced by Nakaoka and Palu ([7]) as a simultaneous generalization of a triangulated category and an exact category.

Let R be a commutative ring and let $\mathcal{C} := (\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an R-linear extriangulated category. For definition and terminologies of extriangulated categories, see [7, 4]. A complex $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ is called an $\mathfrak{s}$-*conflation* if there exists $\delta \in \mathbb{E}(C, A)$ such that $\mathfrak{s}(\delta) = [A \xrightarrow{f} B \xrightarrow{g} C]$, where $[A \xrightarrow{f} B \xrightarrow{g} C]$ is an equivalence class of a complex $A \xrightarrow{f} B \xrightarrow{g} C$. We write the $\mathfrak{s}$-conflation as $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$. Recently, Gorsky, Nakaoka and Palu ([4]) gave an R-bilinear functor $\mathbb{E}^n : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Mod}\, R$ for each $n \geq 2$. We recall examples of extriangulated categories (for detail, see [7, 4])

Example 4. (1) Let $\mathcal{D}$ be a triangulated category with shift functor Σ. Then $\mathcal{D}$ becomes an extriangulated category by the following data.
- $\mathbb{E}(C, A) := \mathcal{D}(C, \Sigma A)$ for all $A, C \in \mathcal{D}$.
- For $\delta \in \mathbb{E}(C, A)$, we take a triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} \Sigma A$. Then we define $\mathfrak{s}(\delta) := [A \xrightarrow{f} B \xrightarrow{g} C]$.

In this case, we have $\mathbb{E}^k(C, A) = \mathcal{D}(C, \Sigma^k A)$ for all $A, C \in \mathcal{D}$ and $k \geq 1$.

(2) Let $\mathcal{E}$ be an exact category. Then $\mathcal{E}$ becomes an extriangulated category by the following data.
- $\mathbb{E}(C, A) := \mathrm{Ext}^1_{\mathcal{E}}(C, A)$, where $\mathrm{Ext}^1_{\mathcal{E}}(C, A)$ is the set of isomorphism classes of conflations in $\mathcal{E}$ of the form $0 \to A \to B \to C \to 0$ for $A, C \in \mathcal{E}$.
- $\mathfrak{s}$ is the identity.

In this case, we have $\mathbb{E}^k(C, A) = \mathrm{Ext}^k_{\mathcal{E}}(C, A)$ for all $A, C \in \mathcal{D}$ and $k \geq 1$.

For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we define a subcategory ${}^{\perp}\mathcal{X}$ as

$$ {}^{\perp}\mathcal{X} := \{M \in \mathcal{C} \mid \mathbb{E}^k(M, \mathcal{X}) = 0 \text{ for each } k \geq 1\}. $$

Dually, we define a subcategory $\mathcal{X}^{\perp}$. Moreover, the following subcategories play a crucial role in this article.

Definition 5. Let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{C}$.
1. Let $\mathcal{X} * \mathcal{Y}$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $X \to M \to Y \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under extensions* if $\mathcal{X} * \mathcal{X} \subseteq \mathcal{X}$.
2. Let $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $X \to Y \to M \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under cones* if $\mathrm{Cone}(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.
3. Let $\mathrm{Cocone}(\mathcal{X}, \mathcal{Y})$ denote the subcategory of $\mathcal{C}$ consisting of $M \in \mathcal{C}$ which admits an $\mathfrak{s}$-conflation $M \to X \to Y \dashrightarrow$ in $\mathcal{C}$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We say that $\mathcal{X}$ *is closed under cocones* if $\mathrm{Cocone}(\mathcal{X}, \mathcal{X}) \subseteq \mathcal{X}$.
4. We call $\mathcal{X}$ a *thick subcategory* of $\mathcal{C}$ if it is closed under extensions, cones, cocones and direct summands. Let $\mathsf{thick}\mathcal{X}$ denote the smallest thick subcategory containing $\mathcal{X}$.
5. For each $n \geq 0$, we inductively define subcategories $\mathcal{X}^{\wedge}_n$ and $\mathcal{X}^{\vee}_n$ of $\mathcal{C}$ as $\mathcal{X}^{\wedge}_n := \mathrm{Cone}(\mathcal{X}^{\wedge}_{n-1}, \mathcal{X})$ and $\mathcal{X}^{\vee}_n := \mathrm{Cocone}(\mathcal{X}, \mathcal{X}^{\vee}_{n-1})$, where $\mathcal{X}^{\wedge}_{-1} := \{0\}$ and $\mathcal{X}^{\vee}_{-1} := \{0\}$.

Put

$$\mathcal{X}^{\wedge} := \bigcup_{n\geq 0} \mathcal{X}_n^{\wedge}, \quad \mathcal{X}^{\vee} := \bigcup_{n\geq 0} \mathcal{X}_n^{\vee}.$$

When $\mathcal{C}$ is a triangulated category, descriptions of $\mathcal{X}^{\wedge}$ and $\mathcal{X}^{\vee}$ are well-known. Indeed, let $\mathcal{D}$ be a triangulated category (regarded as an extriangulated category) with shift functor Σ. For a subcategory $\mathcal{X}$ and an integer $n \geq 0$, we obtain

$$\mathcal{X}_n^{\wedge} = \mathcal{X} * \Sigma\mathcal{X} * \cdots * \Sigma^n \mathcal{X}.$$

If $\mathcal{X}$ is closed under extensions and a negative shift, then $\mathcal{X}_n^{\wedge} = \Sigma^n \mathcal{X}$ holds. Similarly, if $\mathcal{X}$ is closed under extensions and a positive shift, then $\mathcal{X}_n^{\vee} = \Sigma^{-n}\mathcal{X}$ holds.

We introduce the notion of silting subcategories of an extriangulated category, which is a generalization of silting subcategories of a triangulated category. For a class $\mathcal{X}$ of objects in $\mathcal{C}$, let $\mathsf{add}\mathcal{X}$ denote the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands.

Definition 6. Let $\mathcal{C}$ be an extriangulated category and $\mathcal{M}$ a subcategory of $\mathcal{C}$. We call $\mathcal{M}$ a *silting subcategory* of $\mathcal{C}$ if it satisfies the following conditions.

(1) $\mathcal{M}$ is closed under direct summands.
(2) $\mathbb{E}^k(\mathcal{M}, \mathcal{M}) = 0$ for each $k \geq 1$.
(3) $\mathcal{C} = \mathsf{thick}\mathcal{M}$.

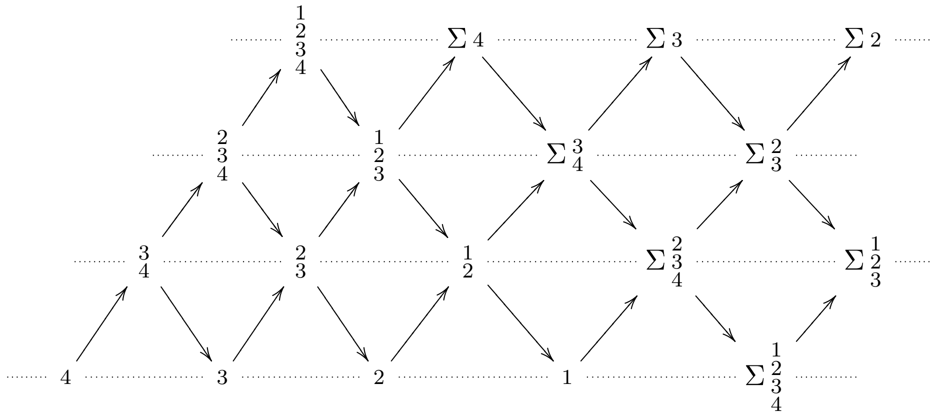
Let $\mathsf{silt}\mathcal{C}$ denote the set of all silting subcategories of $\mathcal{C}$. An object $M \in \mathcal{C}$ is called a *silting object* if $\mathsf{add}M$ is a silting subcategory of $\mathcal{C}$.

We give examples of silting subcategories.

Example 7. (1) Let $\mathcal{D}$ be a triangulated category. Then silting subcategories of a triangulated category $\mathcal{D}$ are exactly silting subcategories of an extriangulated category $\mathcal{D}$.

(2) Let A be an artin algebra and let $\mathcal{P}^{<\infty}(A)$ denote the category of finitely generated right A-modules of finite projective dimension. Since $\mathcal{P}^{<\infty}(A)$ is closed under extensions, it becomes an extriangulated category. We can check that silting objects of $\mathcal{P}^{<\infty}(A)$ coincide with tilting A-modules. Thus if A has finite global dimension, then silting objects of $\mathsf{mod}A$ coincide with tilting A-modules.

Example 8. Let $\mathbf{k}$ be an algebraically closed field. Consider the bounded derived category $\mathcal{D}$ of the path algebra $\mathbf{k}(1 \to 2 \to 3 \to 4)$. Then the Auslander–Reiten quiver of $\mathcal{D}$ is as follows.



Let $\mathcal{X} := \mathsf{add}(\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus 2 \oplus \Sigma 3)$. Since $\mathcal{X}$ is closed under extensions, it follows from [7, Remark 2.18] that $\mathcal{X}$ becomes an extriangulated category. Remark that $\mathcal{X}$ is neither an exact category nor a triangulated category. We can check that $\begin{smallmatrix}3\\4\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus \Sigma 3$ and $\begin{smallmatrix}2\\3\\4\end{smallmatrix} \oplus 2 \oplus \Sigma 3$ are silting objects in $\mathcal{X}$.

We recall the definition of hereditary cotorsion pairs.

Definition 9. Let $\mathcal{C}$ be an extriangulated category and let $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{C}$. We call a pair $(\mathcal{X}, \mathcal{Y})$ a *hereditary cotorsion pair* in $\mathcal{C}$ if it satisfies the following conditions.

(CP1) $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands.
(CP2) $\mathbb{E}^k(\mathcal{X}, \mathcal{Y}) = 0$ for each $k \geq 1$.
(CP3) $\mathcal{C} = \mathrm{Cone}(\mathcal{Y}, \mathcal{X})$.
(CP4) $\mathcal{C} = \mathrm{Cocone}(\mathcal{Y}, \mathcal{X})$.

Let $\mathsf{hcotors}\,\mathcal{C}$ denote the set of hereditary cotorsion pairs in $\mathcal{C}$. For $(\mathcal{X}_1, \mathcal{Y}_1), (\mathcal{X}_2, \mathcal{Y}_2) \in \mathsf{hcotors}\,\mathcal{C}$, we write $(\mathcal{X}_1, \mathcal{Y}_1) \leq (\mathcal{X}_2, \mathcal{Y}_2)$ if $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$. Then $(\mathsf{hcotors}\,\mathcal{C}, \leq)$ clearly becomes a partially ordered set. Remark that if $(\mathcal{X}, \mathcal{Y})$ is a hereditary cotorsion pair in $\mathcal{C}$, then $\mathcal{X}$ is closed under extensions and cocones. Similarly, $\mathcal{Y}$ is closed under extensions and cones.

The following examples show that the notion of hereditary cotorsion pairs in an extriangulated category is a common generalization of co-t-structures on a triangulated category and hereditary cotorsion pairs in an exact category.

Example 10. (1) Let $\mathcal{D}$ be a triangulated category with shift functor Σ. By regarding $\mathcal{D}$ as an extriangulated category, co-t-structures on $\mathcal{D}$ are exactly hereditary cotorsion pairs.

(2) Let $\mathcal{E}$ be an exact category. A pair $(\mathcal{X}, \mathcal{Y})$ of subcategories of $\mathcal{E}$ is called a *hereditary cotorsion pair* in $\mathcal{E}$ if it satisfies the following conditions.
- $\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands.
- $\mathrm{Ext}^k_{\mathcal{E}}(\mathcal{X}, \mathcal{Y}) = 0$ for each $k \geq 1$.
- For each $E \in \mathcal{E}$, there exists a conflation $0 \to Y_E \to X_E \to E \to 0$ such that $Y_E \in \mathcal{Y}$ and $X_E \in \mathcal{X}$.
- For each $E \in \mathcal{E}$, there exists a conflation $0 \to E \to Y^E \to X^E \to 0$ such that $Y^E \in \mathcal{Y}$ and $X^E \in \mathcal{X}$.

By regarding $\mathcal{E}$ as an extriangulated category, hereditary cotorsion pairs in the exact category $\mathcal{E}$ are exactly hereditary cotorsion pairs.

We say that a hereditary cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *bounded* if $\mathcal{C} = \mathcal{X}^\wedge$ and $\mathcal{C} = \mathcal{Y}^\vee$. Let $\mathsf{bdd\text{-}hcotors}\,\mathcal{C}$ denote the partially ordered set of bounded hereditary cotorsion pairs in $\mathcal{C}$. The following theorem is a main result of this article.

Theorem 11 ([1, Theorem 5.7])**.** *Let $\mathcal{C}$ be an extriangulated category. Then there exist mutually inverse bijections*

$$\mathsf{bdd\text{-}hcotors}\,\mathcal{C} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{silt}\mathcal{C},$$

where $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ *and* $\Psi(\mathcal{M}) := (\mathcal{M}^\vee, \mathcal{M}^\wedge) = ({}^\perp\mathcal{M}, \mathcal{M}^\perp)$.

For a triangulated category $\mathcal{D}$, let $\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D}$ denote the set of bounded co-t-structures on $\mathcal{D}$. By regarding $\mathcal{D}$ as an extriangulated category, it follows from Example 10(1) that $\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D} = \mathsf{bdd\text{-}hcotors}\,\mathcal{D}$. Thus we can recover the following result by Theorem 11.

Corollary 12 ([6, Corollary 5.9])**.** *Let $\mathcal{D}$ be a triangulated category. Then there exist mutually inverse bijections*

$$\mathsf{bdd\text{-}co\text{-}t\text{-}str}\,\mathcal{D} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{silt}\mathcal{D},$$

where $\Phi(\mathcal{X}, \mathcal{Y}) := \mathcal{X} \cap \mathcal{Y}$ *and* $\Psi(\mathcal{M}) := (\mathcal{M}^\vee, \mathcal{M}^\wedge)$.

For two subcategories $\mathcal{M}, \mathcal{N}$ of $\mathcal{C}$, we write $\mathcal{M} \geq \mathcal{N}$ if $\mathbb{E}^k(\mathcal{M}, \mathcal{N}) = 0$ for each $k \geq 1$. Since $\mathsf{bdd\text{-}hcotors}\,\mathcal{C}$ is a partially ordered set, the correspondence in Theorem 11 induces a partial order on $\mathsf{silt}\mathcal{C}$.

Corollary 13. *Let $\mathcal{M}, \mathcal{N}$ be silting subcategories of $\mathcal{C}$. Then $\mathcal{M} \geq \mathcal{N}$ if and only if $\mathcal{M}^\wedge \supseteq \mathcal{N}^\wedge$ holds. In particular, $\geq$ gives a partial order on $\mathsf{silt}\mathcal{C}$.*

In the following, we explain that Theorem 11 can recover Auslander–Reiten's result (see Corollary 14). Let $\mathsf{proj}\mathcal{C}$ denote the subcategory of $\mathcal{C}$ consisting of all projective objects in $\mathcal{C}$. We assume that an extriangulated category $\mathcal{C}$ is a Krull–Schmidt category, and has enough projective objects (i.e., $\mathcal{C} = \mathrm{Cone}(\mathcal{C}, \mathsf{proj}\mathcal{C})$) and enough injective objects. For a subcategory $\mathcal{X}$ of $\mathcal{C}$, we call $\mathcal{X}$ a resolving subcategory of $\mathcal{C}$ if $\mathsf{proj}\mathcal{C} \subseteq \mathcal{X}$ and it is closed under extensions, cocones and direct summands. Let $\mathsf{confin\text{-}resolv}\,\mathcal{C}$ denote the set of contravariantly finite resolving subcategories of $\mathcal{C}$. Then there exist mutually inverse bijections

$$\mathsf{hcotors}\,\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{confin\text{-}resolv}\,\mathcal{C},$$

where $F(\mathcal{X}, \mathcal{Y}) = \mathcal{X}$ and $G(\mathcal{X}) = (\mathcal{X}, \mathcal{X}^\perp)$. By restricting these bijections, we have

$$\mathsf{bdd\text{-}hcotors}\,\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \{\mathcal{X} \in \mathsf{confin\text{-}resolv}\,\mathcal{C} \mid \mathcal{X}^\wedge = \mathcal{C}, \mathcal{X} \subseteq (\mathsf{proj}\mathcal{C})^\wedge\}.$$

By Theorem 11, we have mutually inverse bijections

$$\mathsf{silt}\mathcal{C} \underset{\Phi\circ G}{\overset{F\circ\Psi}{\rightleftarrows}} \{\mathcal{X} \in \mathsf{confin\text{-}resolv}\,\mathcal{C} \mid \mathcal{X}^{\wedge} = \mathcal{C}, \mathcal{X} \subseteq (\mathsf{proj}\mathcal{C})^{\wedge}\}.$$

Let A be an artin algebra with finite global dimension. Applying these bijections to $\mathcal{C} = \mathsf{mod}\,A$, we obtain

$$\mathsf{silt}(\mathsf{mod}\,A) \underset{\Phi\circ G}{\overset{F\circ\Psi}{\rightleftarrows}} \mathsf{confin\text{-}resolv}(\mathsf{mod}\,A).$$

Moreover, it follows from Example 7(2) that silting objects of $\mathsf{mod}\,A$ coincide with tilting A-modules. Therefore we have the following result.

Corollary 14 ([2, Corollary 5.6])**.** *Let A be an artin algebra with finite global dimension. Then $T \mapsto {}^{\perp}T$ gives a bijection between the set of isomorphism classes of basic tilting modules and the set of contravariantly finite resolving subcategories, and $T \mapsto T^{\perp}$ gives a bijection between the set of isomorphism classes of basic tilting modules and the set of covariantly finite coresolving subcategories.*

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