APPROXIMATION BY INTERVAL-DECOMPOSABLES AND INTERVAL RESOLUTIONS OF 2D PERSISTENCE MODULES

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Abstract. In topological data analysis, in contrast to the case of one-parameter persistent homology, two-parameter persistent homology presents algebraic difficulties due to its wild representation type. We consider approximations of two-parameter persistence modules: (1) In a previous work, we defined interval approximations using "compression" to essential vertices of intervals together with Möbius inversion. (2) Another idea is to consider homological approximations of persistence modules using interval resolutions. In this work, we first study (2) in the general setting of finite posets and show the following: the interval resolution global dimension is finite for finite posets, and that it can be computed using the Auslander-Reiten translates of the interval representations. Then, in the commutative ladder case, we provide a formula linking the two notions of approximation by a suitable modification of (1). This is an extended abstract summarizing the results of the detailed version [arXiv:2207.03663].

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1. Introduction

In the field of data analysis, one recent development is the rapidly growing subfield called "topological data analysis", which applies ideas from (algebraic) topology for data analysis. One of its main tools is persistent homology [6], which has found applications in various fields of study. Persistent homology is able to describe the topological features (connected components, holes, voids, etc.) of data, and in a multiscale way by providing information of "birth" and "death" parameter values, with respect to one parameter of the data. Algebraically, persistent homology is described as a persistence module, which can be formalized as a representation of an A_n-type quiver, and the topological features are encoded as a choice of generators for an indecomposable decomposition of the persistence module. The indecomposable decomposition is given by interval representations, with the endpoints giving the "birth" and "death" parameter values.

However, coming from motivations in data analysis, there is a need to deal with multiple parameters, leading to multiparameter persistent homology [5]. In this case, the

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underlying parameter space is an n-dimensional commutative grid for n parameters, and the corresponding algebra is of wild representation type for large enough parameter space.

Thus there many attempts to overcome the difficulties in the multiparameter setting, such as by using a suitable generalization of the intervals. The full version [2] of this work studies relative homological algebra with respect to the interval modules, and also gives a more detailed review of the literature. Here, we summarize the results of [2].

2. Background

Throughout, k is a field, vect_k is the category of finite dimensional k-vector spaces, and $\mathcal{P}$ is a finite poset. We first recall the following definitions.

Definition 1.

(1) The *segment* from $p \in \mathcal{P}$ to $q \in \mathcal{P}$ is $[p, q] := \{x \in \mathcal{P} \mid p \leq x \leq q\}$
(2) $\mathcal{P}$ is said to be *connected* if for any $p, q \in \mathcal{P}$, there exists a sequence $p = r_0, r_1, \ldots, r_\ell = q$ of elements with r_{i-1} and r_i comparable for each $i \in \{1, \ldots, \ell\}$.
(3) A subset $S \subseteq \mathcal{P}$ is *convex* if $[p, q] \subseteq S$ for any $p, q \in S$.
(4) A subset $S \subseteq \mathcal{P}$ is an *interval* if it is convex and the subposet induced by S is connected.

We denote by $A := k\mathcal{P}$ the incidence algebra of $\mathcal{P}$ over k. Alternatively, we can consider the Hasse diagram of $\mathcal{P}$ as a quiver Q, and let R be the two-sided ideal of the path algebra of kQ generated by all commutativity relations. With this, we can identify the incidence algebra with the path algebra of the bound quiver (Q, R). We let $\mathrm{mod}\, A$ be the category of finitely generated left A-modules.

Note that a poset $\mathcal{P}$ can be considered as a category with a unique morphism $p \to q$ whenever $p \leq q$. A pointwise finite dimensional (pfd) persistence module over $\mathcal{P}$ is a functor $M : \mathcal{P} \to \mathrm{vect}_k$. Furthermore, $\mathrm{mod}\, k\mathcal{P}$ can be identified with the category of pfd persistence modules over $\mathcal{P}$. We freely identify persistence modules over $\mathcal{P}$, modules over the incidence algebra $k\mathcal{P}$, and representations of the bound quiver (Q, R). In what follows, by persistence module we mean pfd persistence module.

Definition 2. Let $\mathcal{P}$ be a poset and $A = k\mathcal{P}$.

(1) For an interval I of $\mathcal{P}$, the A-module V_I is defined by $V_I(i) = k \ \ (i \in I_0)$, $V_I(i \leq j) = 1_k \ \ (i, j \in I_0)$ and 0 otherwise, is called an *interval module* over A. An A-module M is called *interval-decomposable* if M is isomorphic to a finite direct sum of interval modules.
(2) We denote by $\mathbb{I}(P)$ a set of representatives of all interval modules, with one representative chosen from each isomorphism class. If P is clear we write $\mathbb{I}$.
(3) We denote by $\mathcal{I}(P)$ the set of all interval-decompoable modules. If P is clear we omit it and write $\mathcal{I}$.

We introduce our main poset of interest.

Definition 3 (2D commutative grid)**.** For $m, n \in \mathbb{N} := \{1, 2, \ldots\}$, the poset $\vec{G}_{m,n}$ is defined by

$$\vec{G}_{m,n} = (\{1, \ldots, m\}, \leq) \times (\{1, \ldots, n\}, \leq)$$

and called the $m \times n$ commutative 2D grid. That is, the partial order defined by $(i,j) \leq (k,\ell)$ if and only if $i \leq k$ and $j \leq \ell$.

In topological data analysis, the interval(-decomposable) modules play a central role in one-parameter persistent homology, as they are used to express the "birth" and "death" of topological features. In case of $\mathcal{P} = \vec{G}_{m,n}$, A-modules are called 2-parameter (or 2D) *persistence modules*, and can be used to study the evolution of topological features varying across two parameters. We are interested in approximating 2D persistence modules using interval-decomposable persistence modules.

It is known that each interval I of $\vec{G}_{m,n}$ has a "staircase" form (see the discussion in Section 4.1 of [1]): a full subposet induced by a set of the form

$$I = \{(j,i) \mid i \in \{s, s+1, \dots, t\}, j \in \{b_i, b_i+1, \dots, d_i\}\}$$

for some $1 \leq s \leq t \leq n$ and some $1 \leq b_i \leq d_i \leq m$ for each $s \leq i \leq t$ such that

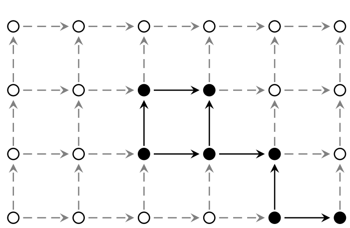
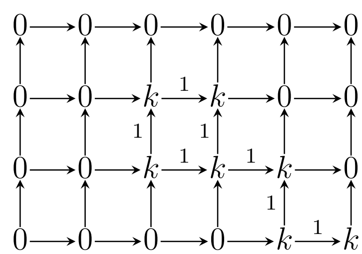
$$b_{i+1} \leq b_i \leq d_{i+1} \leq d_i$$

for all $i \in \{s, \dots, t-1\}$. We adopt the notation of [1] writing

$$I = \bigsqcup_{i=s}^{t} [b_i, d_i]_i$$

to denote the interval above. In this notation, each $[b_i, d_i]_i$ is the "slice" of the staircase at height i.

Example 4. Below is an example of an interval I (filled-in points and arrows) of $\vec{G}_{6,4}$, displaying posets using their Hasse diagrams. This interval is denoted as $[5,6]_1 \sqcup [3,5]_2 \sqcup [3,4]_3$. The corresponding interval module V_I is given to its right.

(2.1) I :  V_I : 

Next, we recall some definitions from relative homological algebra. Throughout the definitions below, $\mathcal{P}$ is a finite poset, $A = k\mathcal{P}$ and M is a persistence module over $\mathcal{P}$.

(1) A *right interval approximation* of M is a morphism $f \in \mathrm{Hom}_A(X, M)$ with $X \in \mathcal{I}$ such that for any $g \in \mathrm{Hom}_A(Y, M)$ with $Y \in \mathcal{I}$ there exists some $h \in \mathrm{Hom}_A(Y, X)$ such that $g = fh$. This is equivalent to saying that f induces an epimorphism

$$\mathrm{Hom}_A(-, f) : \mathrm{Hom}_A(-, X)|_{\mathcal{I}} \to \mathrm{Hom}_A(-, M)|_{\mathcal{I}}.$$

Note that since $\mathcal{I}$ contains all finitely generated projectives, it can be checked that a right interval approximation is guaranteed to be surjective.

(2) An *interval resolution* of M is an exact sequence

$$\cdots \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

such that f_0 is a right interval approximation of M, and for each $i \geq 1$, f_i is a right interval approximation of $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$.

(3) A morphism $f \in \operatorname{Hom}_A(X, M)$ is said to be *right minimal* if every morphism $g\colon X \to X$ with $f = fg$ is an automorphism.

(4) A morphism $f\colon X \to M$ is said to be a *right minimal interval approximation* of M if it is a right interval approximation that is right minimal.

(5) A *minimal interval resolution* of M is an exact sequence

$$\cdots \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

such that f_0 is a right minimal interval approximation of M, and for each $i \geq 1$, f_i is a right minimal interval approximation of $\operatorname{Im} f_i = \operatorname{Ker} f_{i-1}$.

(6) If there exists an interval resolution of M of the form

$$0 \to X_n \xrightarrow{f_n} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0$$

for some $n \geq 0$, we say that *interval resolution dimension* of M is at most n, and write $\operatorname{int-dim} M \leq n$. Otherwise we say that interval resolution dimension of M is infinity.

(7) If $\operatorname{int-dim} M \leq n$ and $\operatorname{int-dim} M \not\leq n-1$, then we say that interval resolution dimension of M is equal to n, and denote it by $\operatorname{int-dim} M = n$.

(8) Finally, we define

$$\operatorname{int-gldim} A := \sup\{\operatorname{int-dim} M \mid M \in \operatorname{mod} A\}$$

and call it the *interval resolution global dimension* of A.

Now, we let

$$G := \bigoplus_{I \in \mathbb{I}(\mathcal{P})} I, \text{ and } \Lambda := \operatorname{End}(G).$$

Since each indecomposable projective module and each indecomposable injective module is isomorphic to some interval in $\mathbb{I}(\mathcal{P})$, G is a generator-cogenerator. Since G is a generator, it is well-known (see for example [4, Proposition 4.17(1)(2)]) that

$$\operatorname{int-dim} M = \operatorname{pd}_\Lambda \operatorname{Hom}_A(G, M) \tag{2.2}$$

where $\operatorname{pd}_\Lambda$ is the projective dimension of Λ-modules. Since G is a generator-cogenerator, we obtain the following equality from Erdmann–Holm–Iyama–Schröer [7, Lemma 2.1]:

$$\operatorname{int-gldim} k\mathcal{P} = \operatorname{gldim} \Lambda - 2. \tag{2.3}$$

3. Results

Proposition 5. *For any finite poset $\mathcal{P}$,* $\operatorname{int-gldim} k\mathcal{P} < \infty$.

The proof of Proposition 5 is provided below. First, the following can be shown.

Lemma 6. *Let M be an interval module of $A = k\mathcal{P}$, and N a submodule of M. Then, N is interval-decomposable.*

Then, we extract the following immediate corollary from known results.

Corollary 7 (Corollary of [11, Theorem in §5], cf. [8, Lemma 2.2])**.** *Let B be an artin algebra, and X a finitely generated B-module. Assume that for each indecomposable direct summand X' of X, all submodules of X' are in* $\operatorname{add} X$*, then* $\operatorname{End}_B(X)$ *is left strongly quasi-hereditary, and its global dimension is finite.*

We remark that the finiteness of the global dimension of $\operatorname{End}_B(X)$ also follows from Corollary 2.4.1(1) together with Theorem 3.3 of [9]. The fact that it is left strongly quasi-hereditary can be shown using [9, Corollary 2.4.1(1)] with [13, Theorem 3.22].

Proof of Proposition 5. This follows immediately by applying Corollary 7 with

$$X := G = \bigoplus_{I \in \mathbb{I}(\mathcal{P})} I, \text{ and } B := \Lambda = \operatorname{End}(G).$$

Note that each indecomposable direct summand X' of X is simply an interval module, and by Lemma 6 each submodule of X' is interval-decomposable and thus is in $\operatorname{add} X$. □

Proposition 8. *For any finite poset $\mathcal{P}$,*

$$\text{int-gldim}\, k\mathcal{P} = \max_{I \in \mathbb{I}} \text{int-dim}(\tau V_I),$$

where τ is the Auslander–Reiten translation.

Proof. Since Λ is not semisimple, we have

$$\begin{aligned} \text{int-gldim}\, k\mathcal{P} &= \operatorname{gldim} \Lambda - 2 \\ &= \max\{\operatorname{pd}(\operatorname{Hom}_A(G, V_I)/\operatorname{rad}(G, V_I)) \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\} - 2 \\ &= \max\{\operatorname{pd} \operatorname{rad}(G, V_I) + 1 \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\} - 2 \\ &= \max\{\operatorname{pd} \operatorname{rad}(G, V_I) - 1 \mid I \in \mathbb{I}, \operatorname{rad}(G, V_I) \neq 0\} \end{aligned}$$

where the first equality is Eq. (2.3). For V_I is not projective, there exists an almost split sequence of the form $0 \to \tau V_I \to E_I \to V_I \to 0$, yielding an exact sequence $0 \to \operatorname{Hom}_A(G, \tau V_I) \to \operatorname{Hom}_A(G, E_I) \to \operatorname{rad}(G, V_I) \to 0$ of Λ-modules, showing that

$$\operatorname{pd} \operatorname{rad}(G, V_I) \leq \max\{\operatorname{pd} \operatorname{Hom}_A(G, E_I), \operatorname{pd} \operatorname{Hom}_A(G, \tau V_I) + 1\}.$$

Applying, we obtain

$$\begin{aligned} \text{int-gldim}\, k\mathcal{P} &\leq \max\{\operatorname{pd} \operatorname{Hom}_A(G, E_I) - 1, \operatorname{pd} \operatorname{Hom}_A(G, \tau V_I)\} \mid I \in \mathbb{I}\} \\ &= \max\{\text{int-dim}\, E_I - 1, \text{int-dim}\, \tau V_I \mid I \in \mathbb{I}\} \\ &\leq \text{int-gldim}\, k\mathcal{P}. \end{aligned}$$

where the second line follows from Eq. (2.2).

By Proposition 5, int-gldim $k\mathcal{P} = d$ for some positive integer d, and hence there exists some $I \in \mathbb{I}$ such that either int-dim $E_I - 1 = d$ or int-dim $\tau V_I = d$. In the former case, we have int-dim $E_I = d + 1 >$ int-gldim $k\mathcal{P} = \max\{\text{int-dim}\, X \mid X \in \operatorname{mod} k\mathcal{P}\}$, a contradiction. Therefore, d is the maximum of $\{\text{int-dim}\, \tau V_I \mid I \in \mathbb{I}\}$. □

We use Proposition 8 in computational experiments, and obtain some conjectures about the value of int-gldim for the 2D commutative grids.

Example 9. Let $k = \mathbb{F}_2$, the finite field with 2 elements, and $A = k\vec{G}_{m,n}$ $(m, n \geq 2)$. In the table below, the row labelled n and column labelled m contains the value (or a lower bound) of int-gldim $k\vec{G}_{m,n}$ obtained by numerical computation.

	2	3	4	5	6	7	8	9	10
2	0	1	2	2	2	2	2	2	2
3	1	2	3	4	4	4			
4	2	3	4	5	≥ 6	≥ 6			
5	2	4	5						
6	2	4	≥ 6						
7	2	4	≥ 6						

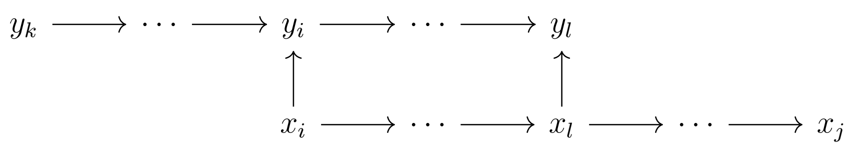
We conjecture that for the row $n = 2$, the value of int-gldim $k\vec{G}_{m,2}$ is 2 for all $m \geq 4$. We further conjecture that for each fixed row n, the value of int-gldim $k\vec{G}_{m,n}$ eventually stabilizes to some fixed constant $C(n)$, and that this happens for $m \geq n + 2$.

The commutative ladder $\vec{G}_{m,2}$ case

From here on, we consider only the commutative ladder, that is, the $m \times 2$ commutative grid $\vec{G}_{m,2}$ (or symmetrically, $\vec{G}_{2,n}$).

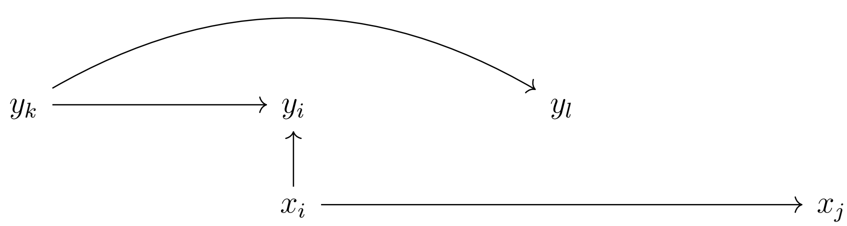
Let us fix some notation and let $I = [x_i, x_j]_1 \sqcup [y_k, y_l]_2$ be an interval of $\vec{G}_{m,2}$. Thus, $1 \leq k \leq i \leq l \leq j \leq n$, and I is illustrated by its Hasse diagram

(3.1)



Given a persistence module M of $\vec{G}_{2,n}$, we "compress" M using I in the following way. Let S_I be the subposet of I with the following Hasse diagram:

(3.2)



Note that S_I is *not* a full subposet of I, since for example $y_i < y_l$ in I but $y_i \not< y_l$ in S_I. Viewing posets as categories, we define the inclusion functor $\xi(I) : S_I \to \vec{G}_{2,n}$, one for each $I \in \mathbb{I}$. We then define $R_{\xi(I)}(M) := M \circ \xi(I)$, which is M restricted ("compressed") to S_I.

Definition 10 (Compressed multiplicity)**.** The *compressed multiplicity with respect to* ξ of V_I in M is

$$c_M^{\xi}(I) := d_{R_{\xi(I)}(M)}(R_{\xi(I)}(V_I)),$$

the multiplicity of $R_{\xi(I)}(V_I)$ as a direct summand of $R_{\xi(I)}(M)$.

Note that there are other ways of "compressing" M, which can be set by changing the choice of the functor $\xi(I)$ (and S_I). In fact, the definition here is a modification of the compressed multiplicity introduced in [3], where the functor is defined as an inclusion of the full subposet of a set of "essential vertices" of I. As another example, when the functor is defined using the inclusion of $I \hookrightarrow \vec{G}_{2,n}$ as is, one recovers the generalized rank invariant of Kim and Memoli [10] (see [3] for a detailed discussion).

The following theorem relates the compressed multiplicity c_M^ξ in terms of a formula involving only the multiplicities of the intervals in an interval resolution of M.

Theorem 11. *Let M be a persistence module over $\vec{G}_{2,n}$ with an interval resolution*

$$0 \to X_r \xrightarrow{f_r} \cdots \to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \to 0, \tag{3.3}$$

with each term X_i a direct sum of interval modules V_J as $X_i \cong \bigoplus_{J\in\mathbb{I}} V_J^{d_J^{(i)}}$. Then,

$$c_M^\xi(I) = \sum_{I\subseteq J\in\mathbb{I}} \sum_{i=0}^{r} (-1)^i d_J^{(i)}.$$

Proof. See the detailed version [2] for a proof. □

The set of intervals $\mathbb{I}(\vec{G}_{m,2})$ can be given a poset structure with partial order defined by $I \leq J$ if and only if I is a subposet of J. Following previous works [10, 3], we use Möbius inversion [12] of c_M^ξ viewed as a function on the elements of $\mathbb{I}(\vec{G}_{m,2})$, to obtain another invariant for persistence modules.

Definition 12 (Interval approximation)**.** The *interval approximation δ_M^ξ with respect to ξ of M* is the Möbius inversion of c_M^ξ:

$$\delta_M^\xi(J) := \sum_{S\subseteq \mathrm{Cov}(J)} (-1)^{\#S} c_M^\xi(\bigvee S)$$

for all $J \in \mathbb{I}$, where $\mathrm{Cov}(J)$ is the set of "cover elements" of J, and $\bigvee S$ is the join of the elements of S (see [3] for a detailed discussion of the poset structure of $\mathbb{I}(\vec{G}_{m,n})$).

Applying Möbius inversion to Theorem 11, we immediately obtain the following.

Corollary 13. *Let M be a persistence module over $\vec{G}_{2,n}$ with an interval resolution as in Theorem 11. Then we have*

$$\delta_M^\xi(J) = \sum_{i=0}^{r} (-1)^i d_J^{(i)}$$

for all $J \in \mathbb{I}$.

This links the two notions of "approximation": one is combinatorial via Möbius inversion (δ_M^ξ), and the other is coming from relative homological algebra (the multiplicities of the intervals in an interval resolution of M).

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