A CHARACTERIZATION OF STANDARD DERIVED EQUIVALENCES OF DIAGRAMS OF DG CATEGORIES AND THEIR GLUING

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Abstract. A diagram consisting of differential graded (dg for short) categories and dg functors is formulated as a colax functor X from a small category I to the 2-category $\mathbb{k}$-**dgCat** of small dg categories, dg functors and dg natural transformations for a fixed commutative ring $\mathbb{k}$. If I is a group regarded as a category with only one object $*$, then X is nothing but a colax action of the group I on the dg category $X(*)$. In this sense, this X can be regarded as a generalization of a dg category with a colax action of a group. We define a notion of standard derived equivalence between such colax functors by generalizing the corresponding notion between dg categories with a group action. Our first main result gives some characterizations of this notion without an assumption of $\mathbb{k}$-flatness (or $\mathbb{k}$-projectivity) on X, one of which is given in terms of generalized versions of a tilting object and a quasi-equivalence. On the other hand, for such a colax functor X, the dg categories $X(i)$ with i objects of I can be glued together to have a single dg category $\int X$, called the Grothendieck construction of X. Our second main result insists that for such colax functors X and X', the Grothendieck construction $\int X'$ is derived equivalent to $\int X$ if there exists a standard derived equivalence from X' to X. These results generalize the main results of [3] and [4] to the dg case, respectively. These are new even for dg categories with a group action. In particular, the second result gives a new tool to show the derived equivalence between the orbit categories of dg categories with a group action (see [6] for such examples).

Key Words: derived equivalence, dg category, Grothendieck construction, 2-category, colax functor
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1. Introduction

Throughout this note $\mathbb{k}$ is a commutative ring, and I is a small category. In [2], when $\mathbb{k}$ is an algebraically closed field, we classified (basic, connected) representation-finite selfinjective algebras up to derived equivalences. They are divided into two classes: the class sRFS of standard algebras and the class nRFS of nonstandard algebras. The sRFS forms a major part. We denote by sRFS$'$ to be the subclass of sRFS consisting of algebras not isomorphic to $\mathbb{k}$. Then sRFS is a disjoint union of sRFS$'$ and the derived equivalence class of $\mathbb{k}$ that coincides with the isoclass of $\mathbb{k}$. We here review how sRFS$'$ was classified. Each member A of sRFS$'$ has the form $\hat{B}/G$ of the orbit category, where B is a tilted

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algebra of Dynkin type Δ, and $\hat{B}$ is the repetitive category of B having a G-action with G an infinite cyclic group. Thus there exists a G-covering $P\colon \hat{B} \to A$. Then we defined the *derived equivalence type* $\operatorname{typ}(A) := (\Delta, f, t)$ ($f \in \mathbb{Q}, t \in \{1, 2, 3\}$) of A, where f, t were derived from the information of the action of a generator of G, and the $\operatorname{typ}(A)$ was shown to be derived invariant of A. In addition, from each type T in the list of all possible types, the normal form $\Lambda(T)$ was constructed. Let A' be another member of sRFS$'$ with a G-covering $P'\colon \hat{B}' \to A'$, $A' \cong \hat{B}/G$, and $\operatorname{typ}(A') = (\Delta', f', t')$. To classify sRFS$'$, it is enough to show that A and A' are derived equivalent if and only if $\operatorname{typ}(A) = \operatorname{typ}(A')$. The only if part means that $\operatorname{typ}(A)$ is derived invariant. The if part is proved by showing that A is derived equivalent to $\Lambda(\operatorname{typ}(A))$. (Thus we may assume that $A' = \Lambda(\operatorname{typ}(A))$.) Note that if $\operatorname{typ}(A) = \operatorname{typ}(A')$, then since $\Delta = \Delta'$, both B and B' are derived equivalent to the hereditary algebra $\mathbb{k}Q$ with Q a Dynkin quiver of type Δ, and hence B and B' are derived equivalent. Then the main tools for the proof of if part were as follows given in [1]:

(1) If B and B' are derived equivalent, then so are $\hat{B}$ and $\hat{B}'$.
(2) If $\hat{B}$ and $\hat{B}'$ are derived equivalent satisfying an additional compatibility condition with P, P', then $\hat{B}/G$ and $\hat{B}'/G$ are derived equivalent.

The tool (2) is generalized in [3, 4] as follows. First, the setting is changed as follows. The algebraically closed field $\mathbb{k}$ is changed to any commutative ring. The cyclic group G is regarded as a category with single object $*$, and is changed to a small category I. $\hat{B}$ is changed to any small $\mathbb{k}$-category $\mathscr{C}$. The G-action $G \to \operatorname{Aut}(\mathscr{C})$ on $\mathscr{C}$ is regarded as a functor X from G as a category with single object $*$ to the category of small $\mathbb{k}$-categories with $X(*) = \mathscr{C}$, and the G-action on $\mathscr{C}$ is changed to a colax functor $X\colon I \to \mathbb{k}\text{-}\mathbf{Cat}$ (see Example 5). The "derived module categoy" $\mathscr{D}(\operatorname{Mod} X)$ is defined as a colax functor from I to the 2-category $\mathbb{k}\text{-}\underline{\mathbf{TRI}}^2$ of triangulated 2-moderate[1] categories, and X is defined to be derived equivalent to another colax functor $X'\colon I \to \mathbb{k}\text{-}\mathbf{Cat}$ if $\mathscr{D}(\operatorname{Mod} X)$ and $\mathscr{D}(\operatorname{Mod} X')$ are equivalent in the 2-category of colax functors $I \to \mathbb{k}\text{-}\underline{\mathbf{TRI}}^2$. The orbit category $\mathscr{C}/G$ for a category $\mathscr{C}$ with a G-action $X \in \operatorname{Aut}(\mathscr{C})$ is changed to the Grothendieck construction $\int X$. In this general setting, the following two questions arise to generalize the tool (2).

Q 1. Characterize derived equivalence for X and X'.
Q 2. When $\int X$ and $\int X'$ are derived equivalent?

Answers are given as the following two theorems.

Theorem 1. *Let $X, X'\colon I \to \mathbb{k}\text{-}\mathbf{Cat}$ be colax functors. Then* (1) *implies* (2)*:*

(1) *X' is derived equivalent to X.*
(2) *X' is equivalent to a tilting colax functor[2] $\mathscr{T}$ for X.*

If X is $\mathbb{k}$-flat[3], then the converse holds.

Theorem 2. *If* (2) *above holds, then $\int X'$ is derived equivalent to $\int X$.*

Since characterization of derived equivalences of $\mathbb{k}$-categories are well controlled in the setting of dg categories as in Keller [8], it is interesting to generalize these theorems to

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[1] See Definition 3.
[2] This is defined in a way similar to Definition 14(2).
[3] The $\mathbb{k}$-modules $X(i)(x, y)$ are flat for all objects i of I and objects x, y of $X(i)$.

dg categories. In this connection, the purpose of the talk is to give a characterization of standard derived equivalences of colax functors from I to the 2-category of dg categories, and to extend Theorem 2 in this setting.

2. Preliminaries

In this section, we collect necessary terminologies.

2.1. **A set theory for the foundation of category theory.** First of all, we remark the set theoretical foundation that we use here, which is needed because we collect many categories. We refer the reader to [5, Appendix A] for details. To avoid the set theoretic paradox, it is usually enough to consider three kinds of collections: sets, classes, and conglomerates. However, to construct mathematical theory only within the scope of sets, one considers a (Grothendieck) universe $\mathfrak{U}$, and assume the *axiom of universes* stating that any set is an element of a universe. We note that the class of all universes is well-ordered. An element of $\mathfrak{U}$ is called a $\mathfrak{U}$-small set, and a subset of $\mathfrak{U}$ is called a $\mathfrak{U}$-class. If a collection S constructed from $\mathfrak{U}$-sets and $\mathfrak{U}$-classes cannot be a $\mathfrak{U}$-class, for example a set of the form $\mathscr{D}(\operatorname{Mod} A)(X,Y)$, where A is a $\mathfrak{U}$-small algebra, X, Y are objects of the derived category of the $\mathfrak{U}$-small modules over A, then we take the smallest universe $\mathfrak{U}'$ having S as its element, and we next consider $\mathfrak{U}'$-small sets, and $\mathfrak{U}'$-classes. If we repeat this procedure, we need more and more universes. To avoid this repetition, we adopt the hierarchy proposed by Levy [9]. First we fix a universe $\mathfrak{U}$ once for all, and we construct mathematical theory within $\mathfrak{U}'$-small sets, where $\mathfrak{U}'$ is the smallest universe having the power set of $\mathfrak{U}$ as its element. In particular, all categories discussed here are small categories in the usual sense. For a category $\mathscr{C}$, the set of all objects of $\mathscr{C}$ is denoted by $\mathscr{C}_0$. Levy's hierarchy defines a $\mathfrak{U}'$-small set $\mathbf{Class}_0^k$ of the k-classes for each non-negative integer k, and we have a sequence of strict inclusions

$$\mathbf{Class}_0^0 \subset \mathbf{Class}_0^1 \subset \mathbf{Class}_0^2 \subset \cdots,$$

where 0-classes are nothing but $\mathfrak{U}$-small sets, usually called just as small sets, and 1-classes are nothing but $\mathfrak{U}$-classes. See [5, Definition A.2.5] for definition of $\mathbf{Class}_0^k$ for $k \geq 2$.

Definition 3. Let $\mathscr{C}$ be a category.

1. $\mathscr{C}$ is called a *small* category[4] if $\mathscr{C}_0$ and $\mathscr{C}(x,y)$ are small for all $x, y \in \mathscr{C}_0$.
2. $\mathscr{C}$ is called a *light* category if $\mathscr{C}_0$ is a 1-class, and $\mathscr{C}(x,y)$ are small for all $x, y \in \mathscr{C}_0$.
3. For each $k \geq 1$, $\mathscr{C}$ is called a k-*moderate* category if $\mathscr{C}_0$ and $\mathscr{C}(x,y)$ are k-classes for all $x, y \in \mathscr{C}_0$.

2.2. **2-categories and colax functors.**

Definition 4. A 2-*category* $\mathbf{C}$ is a sequence of data:

1. a non-empty set $\mathbf{C}_0$,
2. a family of categories $(\mathbf{C}(x,y))_{x,y\in\mathscr{C}_0}$,
3. a family of functors $\circ := (\circ_{x,y,z} : \mathbf{C}(y,z) \times \mathbf{C}(x,y) \to \mathbf{C}(x,z))_{x,y,z\in\mathbf{C}_0}$, and
4. a family of functors $(u_x : \mathbf{1} \to \mathbf{C}(x,x))_{x\in\mathbf{C}_0}$, where $\mathbf{1}$ is the category consisting of one object $*$ and one morphism 1_* (we set $\mathbb{1}_x := u_x(*)$, $\mathbb{1}_{\mathbb{1}_x} := u_x(\mathbb{1}_*)$)

[4] abbreviation of a $\mathfrak{U}$-small category

that satisfies associativity and unitality.

Elements of $\mathbf{C}_0$ are called *objects* of $\mathbf{C}$, elements of $\mathbf{C}_1 := \bigcup_{x,y\in\mathbf{C}_0} \mathbf{C}(x,y)_0$ are called 1-*morphisms* of $\mathbf{C}$, and elements of $\mathbf{C}_2 := \bigcup_{x,y\in\mathbf{C}_0} \mathbf{C}(x,y)_1$ are called 2-*morphisms* of $\mathbf{C}$. The compositions in $\mathbf{C}(x,y)$ with $x,y\in\mathbf{C}_0$ are called *vertical compositions* of 2-morphisms, and the composition $\circ$ for 2-morphisms are called *horizontal compositions*. Sometimes 2-categories are defined by giving the set of objects, 1-morphisms and 2-morphisms, and by omitting the definitions of vertical and horizontal compositions and identities, when they are obvious.

Example 5 (2-categories)**.**

(1) We denote by $\Bbbk$-**Cat** the 2-category of small categories, functors between them, and natural transformations between those functors. Similarly, $\Bbbk$-**FRB**, $\Bbbk$-**TRI** and $\Bbbk$-$\underline{\mathbf{TRI}}^k$ denote the 2-category of light Frobenius $\Bbbk$-categories, of light triangulated $\Bbbk$-categories and of k-moderate triangulated $\Bbbk$-categories ($k \ge 1$), respectively.

(2) Any category $\mathscr{C}$ can be regarded as a 2-category $\mathscr{C}'$ defined as follows, and we identify $\mathscr{C}$ with $\mathscr{C}'$ below, especially for $\mathscr{C} = I$: Objects of $\mathscr{C}'$ are the objects of $\mathscr{C}$; 1-morphisms in $\mathscr{C}'$ are the morphisms in $\mathscr{C}$; and 2-morphisms in $\mathscr{C}'$ are the identities $\mathbb{1}_f$ with $f \in \mathscr{C}(x,y)$ for all $x,y\in\mathscr{C}_0$.

Definition 6. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. A 2-*functor* $X : \mathbf{A} \to \mathbf{B}$ is a pair of data:

(1) a map $X_0 : \mathbf{A}_0 \to \mathbf{B}_0$ (we set $X(x) := X_0(x)$ for all $x \in \mathbf{A}_0$ for short), and

(2) a family of functors $(X_{(x,y)} : \mathbf{A}(x,y) \to \mathbf{B}(X(x),X(y)))_{x,y\in\mathbf{A}_0}$ (we set $X(f) := X_{(x,y)}(f)$ for all $f \in \mathbf{A}(x,y)$ for short)

that preserves compositions and identities.

Definition 7. Let $\mathbf{A}$ and $\mathbf{B}$ be 2-categories. A *colax functor* from $\mathbf{A}$ to $\mathbf{B}$ is a quadruple of data:

(1) , (2) as above,

(3) a family $(X_i)_{i\in\mathbf{A}_0}$ of 2-morphisms $X_i \colon X(\mathbb{1}_i) \Rightarrow \mathbb{1}_{X(i)}$ in $\mathbf{B}$ indexed by $i \in \mathbf{A}_0$, and

(4) a family $(X_{b,a})_{(b,a)}$ of 2-morphisms $X_{b,a} \colon X(ba) \Rightarrow X(b)X(a)$ in $\mathbf{B}$ indexed by $(b,a) \in \mathrm{com}(\mathbf{A}) := \{(b,a) \in \mathbf{A}_1 \times \mathbf{A}_1 \mid ba \text{ is defined}\}$

that satisfies the axioms

(a) Counitality: For each $a \colon i \to j$ in $\mathbf{A}_1$ the following are commutative:

$$\begin{array}{ccc} X(a\mathbb{1}_i) & \xRightarrow{X_{a,\mathbb{1}_i}} & X(a)X(\mathbb{1}_i) \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle X(a)X_i} \\ & & X(a)\mathbb{1}_{X(i)} \end{array} \quad\text{and}\quad \begin{array}{ccc} X(\mathbb{1}_j a) & \xRightarrow{X_{\mathbb{1}_j,a}} & X(\mathbb{1}_j)X(a) \\ & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle X_j X(a)} \\ & & \mathbb{1}_{X(j)}X(a) \end{array} \;;\ \text{and}$$

(b) Coassociativity: For each $i \xrightarrow{a} j \xrightarrow{b} k \xrightarrow{c} l$ in $\mathbf{A}_1$ the following is commutative:

$$\begin{array}{ccc} X(cba) & \xRightarrow{X_{c,ba}} & X(c)X(ba) \\ {\scriptstyle X_{cb,a}}\Downarrow & & \Downarrow{\scriptstyle X(c)\theta_{b,a}} \\ X(cb)X(a) & \xRightarrow[X_{c,b}X(a)]{} & X(c)X(b)X(a). \end{array}$$

A *pseudofunctor* is a colax functor with all X_i and $X_{b,a}$ 2-isomorphisms. A 2-*functor* is nothing but a colax functor with all X_i and $X_{b,a}$ identities.

2.3. **Dg categories.** We now review necessary terminologies for dg categories.

Definition 8 (Dg categories and dg functors)**.**

(1) A *dg category* (a short form of *differential graded category*) is a $\Bbbk$-category $\mathscr{A}$ whose morphism spaces $\mathscr{A}(x,y)$ are (cochain) complexes of $\Bbbk$-modules for all $x, y \in \mathscr{A}_0$, and whose compositions

$$\mathscr{A}(y,z) \otimes_{\Bbbk} \mathscr{A}(x,y) \to \mathscr{A}(x,z)$$

are chain maps of complexes for all $x, y, z \in \mathscr{A}_0$. Note that the Leibniz rule is automatically satisfied.

(2) Let $\mathscr{A}, \mathscr{B}$ be dg categories. Then a *dg functor* $F\colon \mathscr{A} \to \mathscr{B}$ is a sequence of data
  (a) a map $F_0\colon \mathscr{A}_0 \to \mathscr{B}_0$, where we set $F(x) := F_0(x)$ for all $x \in \mathscr{A}_0$ fir short; and
  (b) a family $(F_{(x,y)}\colon \mathscr{A}(x,y) \to \mathscr{B}(F(x), F(y)))_{(x,y)\in\mathscr{A}_0\times\mathscr{A}_0}$ of chain maps, where we set $F(f) := F_{(x,y)}(f)$ for all $f \in \mathscr{A}(x,y)$ for short;

that preserves compositions and identities.

Definition 9. Let $\mathscr{A}, \mathscr{B}$ be dg categories, $E, F\colon \mathscr{A} \to \mathscr{B}$ dg functors, and $n \in \mathbb{Z}$. Then we set $\mathscr{H}om(E,F)^n$ to be the set of all $(\alpha_x^n)_{x\in\mathscr{A}_0} \in \prod_{x\in\mathscr{A}_0} \mathscr{B}(E(x), F(x))^n$ such that $F(f)\alpha_x^n = (-1)^{mn}\alpha_y^n E(f)$ for all $f \in \mathscr{A}(x,y)^m, m \in \mathbb{Z}, x, y \in \mathscr{A}_0$. Using this we define a complex $\mathscr{H}om(E,F) := \bigoplus_{n\in\mathbb{Z}} \mathscr{H}om(F,G)^n$ of $\Bbbk$-modules with the differential d given by $\mathscr{H}om(E,F)^n \to \mathscr{H}om(E,F)^{n+1}, (\alpha_x^n)_x \mapsto (d_{\mathscr{B}}(\alpha_x^n))_x$. Then $\alpha^n := (\alpha_x^n)_{x\in\mathscr{A}_0}$ is called a *derived transformation of degree* n, and $\alpha := (\alpha^n)_{n\in\mathbb{Z}}$ is called a *derived transformation*. An element α of $Z^0(\mathscr{H}om(E,F))$ is called a *dg natural transformation*, which is identified with the family $\alpha := (\alpha_x)_x \in \prod_{x\in\mathscr{A}_0} \mathscr{B}(E(x), F(x))^0$ with $d(\alpha) = 0$, and $F(f)\alpha_x = \alpha_y E(f)$ for all $f \in \mathscr{A}(x,y)$.

Definition 10. By $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ we denote the category of (co)chain complexes of $\Bbbk$-modules, where for any complexes M, N the morphism space is given by

$$\mathscr{C}_{\mathrm{dg}}(\Bbbk)(M,N) := \bigoplus_{n\in\mathbb{Z}} \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\Bbbk}(M^p, N^{p+n})$$

with the differential d defined by $d(f) := (d_N^{p+n} f^p - (-1)^n f^{p+1} d_M^d)_{p\in\mathbb{Z}}$ for all $f = (f^p)_{p\in\mathbb{Z}} \in \mathscr{C}_{\mathrm{dg}}(M,N)^n$. Then $\mathscr{C}_{\mathrm{dg}}(\Bbbk)$ is a light dg category.

We denote by $\Bbbk$-**dgCat** the 2-category of *small* dg categories, dg functors between them, and *dg natural transformations* between those dg functors. By changing small/light or dg natural/derived transformations, we have the following four variants:

	dg natural	derived
small	$\Bbbk$-**dgCat**	$\Bbbk$-**DGCat**
light	$\Bbbk$-**dgCAT**	$\Bbbk$-**DGCAT**

Let $\mathscr{A} \in \Bbbk\text{-}\mathbf{dgCat}_0 = \Bbbk\text{-}\mathbf{DGCat}_0 \ni \mathscr{C}_{\mathrm{dg}}(\Bbbk)$. Then we define the dg category

$$\mathscr{C}_{\mathrm{dg}}(\mathscr{A}) := \Bbbk\text{-}\mathbf{DGCat}(\mathscr{A}^{\mathrm{op}}, \mathscr{C}_{\mathrm{dg}}(\Bbbk)) \in \Bbbk\text{-}\mathbf{dgCAT}_0 = \Bbbk\text{-}\mathbf{DGCAT}_0,$$

of dg $\mathscr{A}$-modules, which is a light category. By taking the 0-cocycles, this defines the category $\mathscr{C}(\mathscr{A}) := Z^0(\mathscr{C}_{\mathrm{dg}}(\mathscr{A}))$ of dg $\mathscr{A}$-modules, which is a light Frobenius category, the homotopy category $\mathscr{H}(\mathscr{A}) := H^0(\mathscr{C}_{\rm dg}(\mathscr{A}))$ of $\mathscr{A}$, which is equal to the stable category $\underline{\mathscr{C}(\mathscr{A})}$ of $\mathscr{C}(\mathscr{A})$ that is a light triangulated category, and the derived category $\mathscr{D}(\mathscr{A}) := \mathscr{H}(\mathscr{A})[\text{qis}^{-1}]$ of $\mathscr{A}$ as a quotient category of $\mathscr{H}(\mathscr{A})$ with respect to the quasi-isomorphisms (see Definition 11), which is known to be a 2-moderate triangulated category. Then we have $\mathscr{C}_{\rm dg}(\mathscr{A})_0 = \mathscr{C}(\mathscr{A})_0 = \mathscr{H}(\mathscr{A})_0 = \mathscr{D}(\mathscr{A})_0$.

Definition 11. Let $M \in \mathscr{C}_{\rm dg}(\mathscr{A})_0$. A morpism $f\colon M \to N$ in $\mathscr{C}(\mathscr{A})$ is called *quasi-isomorphism* (qis for short) if $H^n(f)\colon H^n(M) \to H^n(N)$ is an isomorphism for all $n \in \mathbb{Z}$. M is said to be *acyclic* if $H^n(M) = 0$ for all $n \in \mathbb{Z}$. M is said to be *homotopically projective* if $\mathscr{H}(\mathscr{A})(M, A) = 0$ for all acyclic complexes $A \in \mathscr{H}(\mathscr{A})$. We set $\mathscr{H}_{\rm p}(\mathscr{A})$ to be the full subcategoryy of $\mathscr{H}(\mathscr{A})$ consisting of homotopically projective objects.

We formulate a diagram of dg categories and dg functors as a colax functor X from I to $\Bbbk$-**dgCat**. We can also regard X as a set of dg categories $X(i)$'s with an action of I, hence as a generalization of a dg category with a group action when I is a group viewed as a category with only one object $*$. For a 2-category $\mathbf{C}$, the colax functors from I to $\mathbf{C}$ also form a 2-category $\operatorname{Colax}(I, \mathbf{C})$ with suitably defined 1-morphisms and 2-morphisms, where a 1-morphism is a pair $(F, \phi)\colon X' \to X$ of a family $F = (F(i)\colon X'(i) \to X(i))_{i\in I_0}$ of 1-morphisms in $\mathbf{C}$, and a family $\phi = (\phi_a\colon X(a)F(i) \Rightarrow F(j)X'(a))_{(a\colon i\to j)\in I_1}$ of 2-morphisms in $\mathbf{C}$.

For a colax functor X in $\operatorname{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$, a dg category $\int X$ is constructed in [6] by "gluing" all dg categories $X(i)$'s together, which is called the *Grothendieck construction* of X, which is nothing but the orbit category $X(*)/G$ when I is a group G.

The correspondence $\mathscr{A} \mapsto \mathscr{C}_{dg}(\mathscr{A})$ can be extended to a pseudofunctor $\mathscr{C}_{dg}\colon \Bbbk\text{-}\mathbf{DGCat} \to \Bbbk\text{-}\mathbf{DGCAT}$. Similarly, we obtain pseudofunctors $\mathscr{C}\colon \Bbbk\text{-}\mathbf{dgCAT} \to \Bbbk\text{-}\mathbf{FRB}$, $\mathscr{H}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\mathbf{TRI}$, and $\mathscr{D}\colon \Bbbk\text{-}\mathbf{dgCat} \to \Bbbk\text{-}\underline{\underline{\mathbf{TRI}}}^2$. For a colax functor $X\colon I \to \Bbbk\text{-}\mathbf{dgCat}$, we can define its dg category of dg modules $\mathscr{C}_{dg}(X)$, category of dg modules $\mathscr{C}(X)$, homotopy category $\mathscr{H}(X)$, and derived category $\mathscr{D}(X)$ as the composite $\mathscr{C}_{dg}(X) := \mathscr{C}_{dg} \circ X$ and so on. The relationship of these constructions can be illustrated by the following strict commutative diagram on the left.

$$
\begin{array}{ccccc}
 & & \mathscr{C}_{\rm dg}(\Bbbk\text{-}\mathbf{dgCat}) & & \\
 & \nearrow^{\mathscr{C}_{\rm dg}} & \downarrow Z^0 \quad \searrow^{H^0} & & \\
\Bbbk\text{-}\mathbf{dgCat} & \xrightarrow{\mathscr{C}} & \mathscr{C}(\Bbbk\text{-}\mathbf{dgCat}) & \hookrightarrow & \Bbbk\text{-}\mathbf{FRB} \\
 & \searrow^{\mathscr{H}} & \downarrow \mathrm{st} & & \\
 & & \mathscr{H}(\Bbbk\text{-}\mathbf{dgCat}) & \hookrightarrow & \Bbbk\text{-}\mathbf{TRI} \\
 & \searrow_{\mathscr{D}} & \downarrow \mathbf{L} & & \\
 & & \Bbbk\text{-}\underline{\underline{\mathbf{TRI}}}^2 & &
\end{array}
\quad , \quad
\begin{array}{ccc}
\mathscr{H}(\mathscr{A}) & \xrightarrow{H^0(F)} & \mathscr{H}(\mathscr{B}) \\
\uparrow & & \uparrow \\
\mathscr{H}_{\rm p}(\mathscr{A}) & \xrightarrow{H^0(F)|} & \mathscr{H}_{\rm p}(\mathscr{B}) \\
\mathbf{p}_{\mathscr{A}} \uparrow & & \downarrow \mathbf{j}_{\mathscr{B}} \\
\mathscr{D}(\mathscr{A}) & \dashrightarrow_{\mathbf{L}(F)} & \mathscr{D}(\mathscr{B})
\end{array}
\; (Q_{\mathscr{B}}\colon \mathscr{H}(\mathscr{B}) \to \mathscr{D}(\mathscr{B})) .
$$

Here, in $\mathscr{C}_{\rm dg}(\Bbbk\text{-}\mathbf{dgCat})$, 1-morphisms $F\colon \mathscr{C}_{\rm dg}(\mathscr{A}) \to \mathscr{C}_{\rm dg}(\mathscr{B})$ are required to preserve homotopically projective objects: $F(\mathscr{H}_{\rm p}(\mathscr{A})_0) \subseteq \mathscr{H}_{\rm p}(\mathscr{B})_0$ (similar for $\mathscr{C}(\Bbbk\text{-}\mathbf{dgCat})$ and $\mathscr{H}(\Bbbk\text{-}\mathbf{dgCat})$), which enables us to define a pseudofunctor $\mathbf{L}$ defined by $\mathbf{L}(F) := \mathbf{L}(H^0(F)) = \mathbf{j}_{\mathscr{B}} \circ H^0(F)| \circ \mathbf{p}_{\mathscr{A}}$ in the diagram above, where $Q_{\mathscr{B}}$ is the quotient functor, $\mathbf{j}_{\mathscr{B}}$ is the restriction of $Q_{\mathscr{B}}$ to $\mathscr{H}_{\rm p}(\mathscr{B})$, and $\mathbf{p}_{\mathscr{A}}$ is given by a "projective resolution" with $\mathbf{p}_{\mathscr{A}} \circ \mathbf{j}_{\mathscr{A}} = \mathbb{1}_{\mathscr{H}_{\rm p}(\mathscr{A})}$.

Let $\alpha\colon E \Rightarrow F$ be a dg natural transformation of small dg functors $E, F\colon \mathscr{A} \to \mathscr{B}$ of dg categories. Thus α is a 2-morphism in $\Bbbk$-**dgCat**. We here observe how this α is sent by the pseudofunctors $\mathscr{C}_{\mathrm{dg}}, H^0$ and $\mathbf{L}$. By applying $\mathscr{C}_{\mathrm{dg}}$, we obtain a dg natural transformation $\text{-}\otimes_{\mathscr{A}}\overline{\alpha}\colon \text{-}\otimes_{\mathscr{A}}\overline{E} \Rightarrow \text{-}\otimes_{\mathscr{A}}\overline{F}$ of dg functors $\text{-}\otimes_{\mathscr{A}}\overline{E}, \text{-}\otimes_{\mathscr{A}}\overline{F}\colon \mathscr{C}_{\mathrm{dg}}(\mathscr{A}) \to \mathscr{C}_{\mathrm{dg}}(\mathscr{B})$, where we set $\overline{E}$ to be the $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{B}(\text{-}, E(?))$ (similar for $\overline{F}$), and $\overline{\alpha}$ to be the morphism $\mathscr{B}(\text{-}, \alpha_{(?)})$ of bimodules. This is sent by $\mathbf{L} \circ H^0$ to the natural transformation $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}\colon \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E} \Rightarrow \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}$ of triangle functors $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E}, \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}\colon \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ of derived categories of $\mathscr{A}$ and $\mathscr{B}$, respectively.

3. Results

In this section we state our main results. To state them we need the following three definitions.

Definition 12. Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then X' is said to be *standardly derived equivalent* to X if there exists a 1-morphism $(F, \psi)\colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ in the 2-category $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ such that $\mathbf{L}(F, \psi)\colon\ \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$. Here, this F is said to *preserve homotopically projective objects* if $F(i)(\mathscr{H}_{\mathrm{p}}(X'(i))_0) \subseteq \mathscr{H}_{\mathrm{p}}(X(i))_0$ for all $i \in I_0$.

Remark 13. It is possible to state this sentence using a derived tensor such as: "There exists an X'-X-bimodule Z such that $\text{-}\overset{\mathbf{L}}{\otimes}_{X'}Z\colon \mathscr{D}(X') \to \mathscr{D}(X)$ is an equivalence in $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$." See [6] for details.

Definition 14. Let $\mathscr{A}$ be a small dg category, and $X \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$.

1. A dg subcategory $\mathscr{T}$ of $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ is called a *tilting dg subcategory* for $\mathscr{A}$ if all $T \in \mathscr{T}_0$ is compact and the smallest localizing subcategory of $\mathscr{D}(\mathscr{A})$ containing $\mathscr{T}_0$ coincides with $\mathscr{D}(\mathscr{A})$.
2. A colax subfunctor $\mathscr{T}$ of $\mathscr{C}_{\mathrm{dg}}(X)$ is called a *tilting colax subfunctor* for X if there exists a 1-morphism $(\sigma, \rho)\colon \mathscr{T} \to \mathscr{C}_{\mathrm{dg}}(X)$ such that $\sigma(i)\colon \mathscr{T}(i) \to \mathscr{C}_{\mathrm{dg}}(X(i))$ is the inclusion, and $\mathscr{T}(i)$ is a tilting dg subcategory for $X(i)$ for all $i \in I_0$.

Definition 15 (Quasi-equivalence 1-morphisms)**.**

1. A dg functor $F\colon \mathscr{A} \to \mathscr{B}$ of dg categories is said to be *quasi-equivalence* if $H^n(F)\colon H^n(\mathscr{A}) \to H^n(\mathscr{B})$ is fully faithful for all $n \in \mathbb{Z}$, and $H^0(F)\colon H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an equivalence.
2. A dg natural transformation $\alpha\colon E \Rightarrow F$ of dg functors $E, F\colon \mathscr{A} \to \mathscr{B}$ of dg categories is called a 2-*quasi-isomorphism* if $\text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{\alpha}\colon \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{E} \to \text{-}\overset{\mathbf{L}}{\otimes}_{\mathscr{A}}\overline{F}$ is an isomorphism.
3. A 1-morphism $(F, \phi)\colon X \to \mathscr{T}$ in $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$ is said to be *quasi-equivalence* if $F(i)$ is quasi-equivalence for all $i \in I_0$ and $\phi(a)$ is 2-quasi-isomorphism for all $a \in I_1$.

We obtained the following characterization of standard derived equivalences of diagrams of dg categories, where we do not need $\Bbbk$-flatness assumption unlike a result by Keller [8].

Theorem 16. *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Then the following are equivalent.*

(1) *There exists a* 1*-morphism* $(F,\psi)\colon \mathscr{C}_{\mathrm{dg}}(X') \to \mathscr{C}_{\mathrm{dg}}(X)$ *in* $\mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCAT})$ *such that* F preserves homotopically projective objects *and* $\mathbf{L}(F,\psi)\colon\ \mathscr{D}(X') \to \mathscr{D}(X)$ *is an equivalence in* $\mathrm{Colax}(I, \Bbbk\text{-}\underline{\mathbf{TRI}}^2)$.
(2) *X' is standardly derived equivalent to X.*
(3) *There exists a quasi-equivalence $(E,\phi)\colon X' \to \mathscr{T}$ for some tilting colax functor $\mathscr{T}$ for X.*

Remark 17. The statement (1) guarantees that the relation to be standardly derived equivalent is transitive. But we do not know whether this relation is reflexive.

Remark 18. We do not need $\Bbbk$-flatness assumption on X. It is possible to remove this assumption also from Keller's theorem [8, Corollary 9.2] for dg categories. In connection with this, we mention that Dugger–Shipley [7] proved Rickard's theorem [10, Proposition 5.1] (it needed $\Bbbk$-projectivity) without this assumption.

The following gives a sufficient condition for the Grothendieck constructions to be derived equivalent.

Theorem 19. *Let $X, X' \in \mathrm{Colax}(I, \Bbbk\text{-}\mathbf{dgCat})$. Assume that X' is standardly derived equivalent to X, or equivalently, there exists a quasi-equivalence from X' to a tilting colax functor $\mathscr{T}$ for X (cf. Theorem 16). Then $\int X'$ is derived equivalent to $\int X$.*

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