CONNECTEDNESS OF QUASI-HEREDITARY STRUCTURES

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Abstract. Dlab and Ringel showed that algebras being quasi-hereditary in all orders for indices of primitive idempotents becomes hereditary. So, we are interested in for which orders a given quasi-hereditary algebra is again quasi-hereditary. As a matter of fact, we consider permutations of indices, and if the algebra with permuted indices is quasi-hereditary, then we say that this permutation gives a quasi-hereditary structure.

In this paper, we first give a criterion for adjacent transpositions giving quasi-hereditary structures, in terms of homological conditions of standard or costandard modules over a given quasi-hereditary algebra. Next, we consider those which we call connectedness of quasi-hereditary structures. The definition of connectedness can be found in Definition 4. We then show that any two quasi-hereditary structures are connected, which is our main result. By this result, once we know that there are two quasi-hereditary structures, then permutations in some sense lying between them give also quasi-hereditary structures.

1. Introduction

Quasi-hereditary algebras, introduced by Cline, Parshall and Scott, generalize hereditary algebras. Moreover Dlab and Ringel showed in Theorem 1 of [2] that if an algebra is quasi-hereditary in all orders, it becomes hereditary, and vice versa. From this point of view, we study quasi-hereditary structures for a given algebra. Recently, there are two results on quasi-hereditary structures. Coulembier showed in [1] that a quasi-hereditary algebra with simple preserving duality has only one quasi-hereditary structure. Flores, Kimura and Rognerud gave a method of counting the number of quasi-hereditary structures for a path algebras of Dynkin types in [3]. In their papers, the quasi-hereditary structure was defined by an equivalent class of partial orders with some relations. However in this paper, we define it by using a total order without using equivalent classes. Thus, our results are in the nature different from them and can not be derived from their results. Moreover we will use permutations instead of total orders when considering quasi-hereditary structures.

Throughout this paper, let K be an algebraically closed field, A a finite dimensional K-algebra with pairwise orthogonal primitive idempotents $e_1, \dots, e_n$, and let $\Lambda = \{1, \dots, n\}$. For $i \in \Lambda$, we denote $P(i) = e_iA$ the indecomposable projective module, $S(i)$ the top of $P(i)$, and $I(i)$ the injective envelope of $S(i)$. The standard K-dual $\mathrm{Hom}_K(-, K)$ is denoted by D. For an A-module M, we write the isomorphism class of M by $[M]$ and the Jordan-Hölder multiplicity of $S(i)$ in M by $[M : S(i)]$. Let $\mathfrak{S}_n$ be the symmetric group on n letters, $e \in \mathfrak{S}_n$ the trivial permutation and $\sigma_i = (i, i+1) \in \mathfrak{S}_n$ adjacent transpositions for $1 \leq i \leq n-1$.

The detailed version of this paper will be submitted for publication elsewhere.

First, we recall the definition of quasi-hereditary algebras and quasi-hereditary structures.

Definition 1. Let A be an algebra as above and $\sigma \in \mathfrak{S}_n$.

(1) The total order
$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n)$$
over Λ is called the **$\boldsymbol{\sigma}$-order**.

(2) For each $i \in \Lambda$, the A-module $\Delta^\sigma(i)$, called the **standard module** with respect to the σ-order, is defined by the maximal factor module of $P(i)$ having only composition factors $S(j)$ with $\sigma(j) \leq \sigma(i)$. Moreover we will write the set $\{\Delta^\sigma(1), \ldots, \Delta^\sigma(n)\}$ by Δ^σ.

(3) Dually, the A-module $\nabla^\sigma(i)$, called the **costandard module** with respect to the σ-order, is defined by the maximal submodule of $I(i)$ having only composition factors $S(j)$ with $\sigma(j) \leq \sigma(i)$. Denote the set $\{\nabla^\sigma(1), \ldots, \nabla^\sigma(n)\}$ by ∇^σ.

(4) We say that an A-module M has a **Δ^σ-filtration** (resp. a **∇^σ-filtration**) if there is a sequence of submodules
$$0 = M_{m+1} \subset \cdots \subset M_2 \subset M_1 = M$$
such that for each $1 \leq k \leq m$, $M_k/M_{k+1} \cong \Delta^\sigma(j)$ (resp. $M_k/M_{k+1} \cong \nabla^\sigma(j)$) for some $j \in \Lambda$.

(5) A pair (A, σ) is said to be a **quasi-hereditary algebra** provided that the following conditions are satisfied:
  (a) $[\Delta^\sigma(i) : S(i)] = 1$ for all $i \in \Lambda$.
  (b) A_A has a Δ^σ-filtration.

If this is the case, we say that the permutation σ gives a **quasi-hereditary structure** of A.

Next, we show some properties which every pair of neighbor standard modules has.

Lemma 2 ([5] Lemma 2.)**.** *Assume that (A, e) is a quasi-hereditary algebra. Then we have the following equalities. For $1 \leq i \leq n-1$,*

(1) $\dim \operatorname{Hom}_A(\Delta(i), \Delta(i+1)) = \dim \operatorname{Hom}_A(P(i), \Delta(i+1)) = [\Delta(i+1) : S(i)]$,
(2) $\dim \operatorname{Hom}_A(\nabla(i+1), \nabla(i)) = \dim \operatorname{Hom}_A(\nabla(i+1), I(i)) = [\nabla(i+1) : S(i)]$,
(3) $\dim \operatorname{Ext}^1_A(\Delta(i), \Delta(i+1)) = \dim \operatorname{Ext}^1_A(\Delta(i), S(i+1)) = [P(i) : \Delta(i+1)]$,
(4) $\dim \operatorname{Ext}^1_A(\nabla(i+1), \nabla(i)) = \dim \operatorname{Ext}^1_A(S(i+1), \nabla(i)) = [I(i) : \nabla(i+1)]$.

We will denote
$$H_i = \dim \operatorname{Hom}_A(\Delta(i), \Delta(i+1)),\ E_i = \dim \operatorname{Ext}^1_A(\Delta(i), \Delta(i+1)),$$
$$\overline{H_i} = \dim \operatorname{Hom}_A(\nabla(i+1), \nabla(i)),\ \text{and}\ \overline{E_i} = \dim \operatorname{Ext}^1_A(\nabla(i+1), \nabla(i)).$$

Lemma 3 ([5] Corollary 1)**.** *Assume that (A, e) is a quasi-hereditary algebra. Then the followings hold. For $1 \leq i \leq n-1$,*

(1) $\operatorname{Hom}_A(\Delta(i), \Delta(i+1)) \cong D\operatorname{Ext}^1_A(\nabla(i+1), \nabla(i))$,
(2) $\operatorname{Ext}^1_A(\Delta(i), \Delta(i+1)) \cong D\operatorname{Hom}_A(\nabla(i+1), \nabla(i))$.

In particular, we have $H_i = \overline{E_i}$ and $E_i = \overline{H_i}$.

Finally, we define the connectedness of quasi-hereditary structures, which is the main topic of this paper.

Definition 4. Two permutations σ and τ giving quasi-hereditary structures are said to be **connected** if the following condition holds: There is a decomposition $\tau\sigma^{-1} = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that all $\sigma_{i_k} \cdots \sigma_{i_1}\sigma$ for $1 \leq k \leq l$ also give quasi-hereditary structures. Moreover, if any two permutations giving quasi-hereditary structures are connected, we also say that quasi-hereditary structures are **connected**.

Our aim in this paper is to claim that quasi-hereditary structures are connected.

2. Twistability

Let (A, σ) be a quasi-hereditary algebra. If $(A, \sigma_i\sigma)$ is also quasi-hereditary, then we call the original quasi-hereditary algebra (A, σ) to be i**th-twistable**. In this section, we will give the condition on standard or costandard modules equivalent to the ith-twistability for a quasi-hereditary algebra.

Lemma 5. *Let (A, e) be quasi-hereditary. Then $[\Delta^{\sigma_i}(k) : S(k)] = 1$ for all $k \in \Lambda$ if and only if $E_i\overline{E_i} = 0$.*

By using this lemma, we get a criterion for the ith-twistability.

Theorem 6. *Let (A, e) be quasi-hereditary. Then (A, σ_i) is quasi-hereditary if and only if one of the following conditions holds:*

($\mathcal{E}_i$): *$E_i = 0$ and $\Delta(i+1)$ has a submodule isomorphic to $\Delta(i)^{H_i}$.*

($\overline{\mathcal{E}_i}$): *$\overline{E_i} = 0$ and $\nabla(i+1)$ has a factor module isomorphic to $\nabla(i)^{\overline{H_i}}$.*

In particular, if a quasi-hereditary algebra (A, e) satisfies $E_i = \overline{E_i} = 0$, then (A, σ_i) is also quasi-hereditary with $\Delta^{\sigma_i} = \Delta$ and $\nabla^{\sigma_i} = \nabla$.

3. Connectedness

In this section, we will argue about "connectivity" of quasi-hereditary structures. In general, we can obtain all permutations giving quasi-hereditary structures from one by checking repeatedly whether each quasi-hereditary algebra satisfies the condition ($\mathcal{E}_i$) or ($\overline{\mathcal{E}_i}$). To show the connectedness of quasi-hereditary structures, we first claim that e and another are connected in Theorem 10.

Lemma 7. *Let e, σ give quasi-hereditary structures with $e \neq \sigma$. Then for the minimum element $i \in \Lambda$ satisfying $\sigma(i+1) < \sigma(i)$, it holds that $E_iH_i = 0$.*

Proposition 8. *Let e, σ give quasi-hereditary structures. Then there is a minimal decomposition $\sigma = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that σ_{i_1} gives a quasi-hereditary structure. Here, this i_1 is the element i given in Lemma 7.*

Corollary 9. *Let e, σ give quasi-hereditary structures. Then there is a minimal decomposition $\sigma = \sigma_{i_l} \cdots \sigma_{i_1}$ into the product of adjacent transpositions such that $\sigma_{i_l}\sigma$ gives a quasi-hereditary structure.*

The next theorem is followed from the above corollary and the induction on the length of σ.

Theorem 10. *Let e, σ give quasi-hereditary structures. Then they are connected.*

Finally, by retaking the indices of primitive idempotents, we get the following result.

Theorem 11. *Any two permutations giving quasi-hereditary structures are connected.*

Moreover, for two permutations giving quasi-hereditary structures, we get a sequence of adjacent transpositions which induce the connectedness of them, by Proposition 8. In particular, this sequence is determined by only the permutations and does not depend on the algebra.

Corollary 12. *Let σ, τ give quasi-hereditary structures with $\sigma \neq \tau$. For $k = 1, 2, \dots,$ inductively take a minimal element i_k with respect to the $(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma)$-order satisfying $\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma(i_k) \neq n$ and*

$$\tau(i_k) > \tau\sigma^{-1}\sigma_{i_1} \cdots \sigma_{i_{k-1}}(\sigma_{i_{k-1}} \cdots \sigma_{i_1}\sigma(i_k) + 1).$$

We take $i_1, i_2, \dots, i_k$ until those elements satisfying the above exist. If there is no i_{k+1} satisfying the above, then we do not take i_{k+1} and put $l = k$. Then the product $\sigma_{i_l} \cdots \sigma_{i_1}$ is a decomposition of $\tau\sigma^{-1}$ inducing the connectedness of σ and τ.

Example 13. Consider a quiver $\begin{tikzcd} 1 \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & 2 \arrow[r, "\gamma"] & 3 \arrow[r, "\delta"] & 4 \end{tikzcd}$ and an ideal $I = \langle \alpha\gamma\delta - \beta\delta \rangle$ of KQ, and put $A = KQ/I$. Then all indecomposable projective modules are as follows:

$$P(1) : \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ 3 & & \\ & 4 & \end{smallmatrix},\ P(2) : \begin{smallmatrix} 2 \\ 3 \\ 4 \end{smallmatrix},\ P(3) : \begin{smallmatrix} 3 \\ 4 \end{smallmatrix},\ P(4) : \begin{smallmatrix} 4 \end{smallmatrix}.$$

Now we have 24 permutations on $\Lambda = \{1, 2, 3, 4\}$. In the following, we will write $\sigma_{(i_l \cdots i_2 i_1)} = \sigma_{(i_l, \dots, i_2, i_1)}$ as the product $\sigma_{i_l} \cdots \sigma_{i_2}\sigma_{i_1}$, where $i_k \in \{1, 2, 3\}$ for $1 \leq k \leq l$. For example, the $\sigma_{(21)}$-order is $2 < 3 < 1 < 4$. Let $\lambda = (i_l, \dots, i_2, i_1)$ be a sequence of elements of $\{1, 2, 3\}$ and $\Delta^{(\lambda)}$ be standard modules with respect to the $\sigma_{(\lambda)}$-order.

Clearly $(A, \sigma_{(123121)})$ is quasi-hereditary since all standard modules are projective and satisfy $[P(i) : S(i)] = 1$ for all $i \in \Lambda$. By using $(\mathcal{E}_i)$ in Theorem 6, we have the following diagram which shows that if the source of an arrow gives a quasi-hereditary structure, then so does the target.

$$\begin{tikzcd}
 & \sigma_{(12321)} \arrow[r, "\sigma_1"] & \sigma_{(2321)} \arrow[dr, "\sigma_2"] & & & & \\
\sigma_{(123121)} \arrow[ur, "\sigma_2"] \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_3"'] & \sigma_{(23121)} \arrow[r, "\sigma_2"] \arrow[dr, "\sigma_3"] & \sigma_{(3121)} \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_3"] & \sigma_{(321)} \arrow[r, "\sigma_3"] & \sigma_{(21)} \arrow[r, "\sigma_2"] & \sigma_{(1)} \arrow[r, "\sigma_1"] & e \\
 & \sigma_{(12312)} \arrow[r, "\sigma_1"] & \sigma_{(2312)} \arrow[dr, "\sigma_2"] & \sigma_{(121)} \arrow[ur, "\sigma_1"] \arrow[r, "\sigma_2"] & \sigma_{(12)} \arrow[r, "\sigma_1"] & \sigma_{(2)} \arrow[ur, "\sigma_2"] & \\
 & & & \sigma_{(312)} \arrow[ur, "\sigma_3"] \arrow[r, "\sigma_1"] & \sigma_{(32)} \arrow[ur, "\sigma_3"] & &
\end{tikzcd}$$

However applying Theorem 6 to the quasi-hereditary algebra $(A, \sigma_{(12321)})$, we recognize that it is not 3rd-twistable, i.e., $\sigma_{(1231)}$ does not give a quasi-hereditary structure of

A. Similarly, $(A, \sigma_{(12312)})$ is not 2nd-twistable, and hence $\sigma_{(1232)}$ does not give a quasi-hereditary structure of A. Focus on the two permutations $\sigma_{(1231)}$ and $\sigma_{(1232)}$. Then we finally show that all the other permutations do not give quasi-hereditary structures.

$$
\begin{tikzcd}
\sigma_{(1231)} & \sigma_{(231)} \arrow[l, "\sigma_1"'] & \sigma_{(31)} \arrow[l, "\sigma_2"'] & \sigma_{(3)} \arrow[l, "\sigma_1"'] \\
 & \sigma_{(123)} \arrow[r, "\sigma_1"'] & \sigma_{(23)} \arrow[ur, "\sigma_2"'] & \\
 & & & \\
\sigma_{(1232)} & \sigma_{(232)} \arrow[l, "\sigma_1"'] & &
\end{tikzcd}
$$

In fact, if some permutations in this diagram give quasi-hereditary structures, then $\sigma_{(1231)}$ or $\sigma_{(1232)}$ also does, a contradiction. Hence the permutations in this diagram do not give quasi-hereditary structures of A. Now we complete checking whether each permutation gives a quasi-hereditary structure or not for the algebra A.

References

[1] K. Coulembier, *The classification of blocks in BGG category O*, Mathematische Zeitschrift **295(1)** (2020), 821–837.

[2] V. Dlab, and C.M. Ringel, *Quasi-hereditary algebras*, Illinois Journal of Mathematics **33(2)** (1989), 280–291.

[3] M. Flores, Y. Kimura, and B. Rognerud, *Combinatorics of quasi-hereditary structures*, Journal of Combinatorial Theory, Series A, **187** (2022), 105559.

[4] Y. Goto, *Connectedness of quasi-hereditary structures*, arXiv:2210.17104.

[5] V. Mazorchuk, and S. Ovsienko, *A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra*, Journal of Mathematics of Kyoto University **45(4)** (2005), 711–741.

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