NONCOMMUTATIVE CONICS IN CALABI-YAU QUANTUM PROJECTIVE PLANES I, II

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Abstract. In noncommutative algebraic geometry, noncommutative quadric hypersurfaces are major objects of study. In this paper, we focus on studying the homogeneous coordinate algebras A of noncommutative conics $\operatorname{Proj}_{\mathrm{nc}} A$ embedded into Calabi-Yau quantum projective planes. We give a complete classification of A up to isomorphism of graded algebras. As a consequence, we show that there are exactly 9 isomorphism classes of noncommutative conics $\operatorname{Proj}_{\mathrm{nc}} A$ in Calabi-Yau quantum projective planes.

1. Motivation

Throughout this paper, we fix an algebraically closed field k of characteristic 0. By Sylvester's theorem, it is elementary to classify (commutative) quadric hypersurfaces in $\mathbb{P}^{d-1}$, namely, they are isomorphic to

$$\operatorname{Proj} k[x_1, \dots, x_d]/(x_1^2 + \cdots + x_j^2) \subset \mathbb{P}^{d-1}$$

for some $j = 1, \dots, d$. When $d = 3$, we see that there are exactly 3 isomorphism classes of conics, exactly 1 of them is smooth and exactly 1 of them is irreducible (the same one).

The ultimate goal of our project is to classify noncommutative quadric hypersurfaces in quantum $\mathbb{P}^{d-1}$'s. As a first step to this ultimate goal, we define and classify noncommutative conics in quantum $\mathbb{P}^2$'s.

2. Quantum polynomial algebras

Definition 1 ([1]). A *d-dimensional quantum polynomial algebra* is a connected graded algebra S such that

(1) $\operatorname{gldim} S = d < \infty$,
(2) $\operatorname{Ext}^q_S(k, S) = 0$ if $q \neq d$, and $\operatorname{Ext}^d_S(k, S) \cong k$, and
(3) $H_S(t) = 1/(1-t)^d$.

A d-dimensional quantum polynomial algebra S is a noncommutative analogue of the commutative polynomial algebra $k[x_1, \dots, x_d]$, so the noncommutative projective scheme $\operatorname{Proj}_{\mathrm{nc}} S$ associated to S in the sense of [3] is regarded as a quantum $\mathbb{P}^{d-1}$ (see Section 3 for details).

Next, we recall a notion of geometric algebra for a quadratic algebra.

Definition 2. Let $A = T(V)/(R)$ be a quadratic algebra where V is a finite dimensional vector space and $R \subset V \otimes V$ is a subspace.

The detailed version of this paper will be submitted for publication elsewhere.

(1) A geometric pair (E, σ) consists of a projective scheme $E \subset \mathbb{P}(V^*)$ and an automorphism $\sigma \in \operatorname{Aut} E$.
(2) We say that A satisfies (G1) if there exists a geometric pair (E, σ) such that
$$\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}.$$
In this case, we write $\mathcal{P}(A) = (E, \sigma)$.
(3) We say that A satisfies (G2) if there exists a geometric pair (E, σ) such that
$$R = \{f \in V \otimes V \mid f(p, \sigma(p)) = 0 \ \forall\, p \in E\}.$$
In this case, we write $A = \mathcal{A}(E, \sigma)$.
(4) We say that A is geometric if it satisfies both (G1) and (G2) such that $\mathcal{A}(\mathcal{P}(A)) = A$.

Theorem 3 ([2])**.** *Every 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$ is geometric where either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a cubic divisor.*

Example 4. A typical example of a 3-dimensional quadratic AS-regular algebra is a 3-dimensional Sklyanin algebra
$$k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2) = \mathcal{A}(E, \sigma),$$
where $E = \mathcal{V}((a^3+b^3+c^3)xyz - abc(x^3+y^3+z^3)) \subset \mathbb{P}^2$ is an elliptic curve, and $\sigma \in \operatorname{Aut} E$ is the translation by a point $(a, b, c) \in E$.

3. Quantum projective spaces

Artin and Zhang introduced a notion of noncommutative schemes.

Definition 5 ([3])**.** A *noncommutative scheme* is a pair $X = (\operatorname{mod} X, \mathcal{O}_X)$ consisting of a k-linear abelian category $\operatorname{mod} X$ and an object $\mathcal{O}_X \in \operatorname{mod} X$. We say that two noncommutative schemes X and Y are isomorphic, denoted by $X \cong Y$, if there exists an equivalence functor $F : \operatorname{mod} X \to \operatorname{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

We give some examples of noncommutative schemes.

Example 6. If X is a commutative noetherian scheme, then we view X as a noncommutative scheme by $X = (\operatorname{coh} X, \mathcal{O}_X)$.

Example 7. The *noncommutative affine scheme* associated to a right noetherian algebra R is a noncommutative scheme defined by $\operatorname{Spec}_{\rm nc} R := (\operatorname{mod} R, R)$. If R is commutative, then $\operatorname{Spec}_{\rm nc} R \cong \operatorname{Spec} R$.

Example 8. Let A be a right noetherian connected graded algebra. We define the quotient category $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$ where $\operatorname{tors} A$ is the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k. The *noncommutative projective scheme* associated to A is a noncommutative scheme defined by $\operatorname{Proj}_{\rm nc} A := (\operatorname{tails} A, \pi A)$ where $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ is the quotient functor. If A is commutative and generated in degree 1 over k, then $\operatorname{Proj}_{\rm nc} A \cong \operatorname{Proj} A$.

Definition 9. A *quantum* $\mathbb{P}^{d-1}$ is a noncommutative projective scheme $\operatorname{Proj}_{\rm nc} S$ for some d-dimensional quantum polynomial algebra S.

4. Twisted superpotentials

Definition 10. Let V be a finite dimensional vector space and $m \in \mathbb{N}^+$. Define a linear map $\phi : V^{\otimes m} \to V^{\otimes m}$ by $\phi(v_1 \otimes v_2 \otimes \cdots \otimes v_{m-1} \otimes v_m) = v_m \otimes v_1 \otimes \cdots \otimes v_{m-2} \otimes v_{m-1}$.

1. $w \in V^{\otimes m}$ is called *superpotential* if $\phi(w) = w$.
2. $w \in V^{\otimes m}$ is called *twisted superpotential* if $(\tau \otimes \mathrm{id}^{\otimes m-1})\phi(w) = w$ for some $\tau \in \mathrm{GL}(V)$.
3. The i-th derivation quotient algebra of $w \in V^{\otimes m}$ is defined by $D(w, i) := T(V)/(\partial^i w)$ where $\partial^i w$ is the "i-th left partial derivatives" of w.

The next theorem plays a key role to classify quantum polynomial algebras.

Theorem 11 ([4, Theorem 11])**.** *For every d-dimensional quantum polynomial algebra S, there exists a unique twisted superpotential w such that $S = D(w, d-2)$.*

Example 12. $w = a(xyz + yzx + zxy) + b(xzy + yxz + zyx) + c(x^3 + y^3 + z^3)$ is a superpotential such that

$$D(w, 1) = k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2)$$

is a 3-dimensional Sklyanin algebra.

The next theorem is a characterization of "Calabi-Yau" algebras by using twisted superpotentials.

Theorem 13 ([9, Corollary 4.5])**.** *Let $S = D(w, d-2)$ be a d-dimensional quantum polynomial algebra where w is a twisted superpotential. Then S is "Calabi-Yau" if and only if w is $(-1)^{d+1}$ twisted superpotential.*

Example 14. Every 3-dimensional Sklyanin algebra is Calabi-Yau.

A classification of twisted superpotentials whose derivation-quotient algebras are 3-dimensional quantum polynomial algebras is completed.

Theorem 15 ([10])**.** *Superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.*

Theorem 16 ([7, Theorem 3.4], [8, Theorem 4.2])**.** *Twisted superpotentials w such that $D(w, 1)$ are 3-dimensional quantum polynomial algebras are classified.*

By the above theorem, we have finally completed the Artin-Schelter's project in the quadratic case proposed in [1]. As an application, we have the following.

Theorem 17 ([7, Theorem 4.4])**.** *For every 3-dimensional quantum polynomial algebra S, there exists a 3-dimensional Calabi-Yau quantum polynomial algebra $\tilde{S}$ such that $\mathrm{Proj}_{\mathrm{nc}} S \cong \mathrm{Proj}_{\mathrm{nc}} \tilde{S}$.*

The above theorem tells that every quantum $\mathbb{P}^2$ is isomorphic to a "Calabi-Yau" quantum $\mathbb{P}^2$.

5. Noncommutative conics

In this section, we define a notion of noncommutative quadric hypersurface in a quantum $\mathbb{P}^{d-1}$.

Definition 18. A *noncommutative quadric hypersurface in a (Calabi-Yau) quantum* $\mathbb{P}^{d-1}$ is the noncommutative projective scheme $\mathrm{Proj}_{\mathrm{nc}} S/(f)$ for some d-dimensional (Calabi-Yau) quantum polynomial algebra S and for some regular central element $f \in Z(S)_2$. In particular, when $d = 3$ (resp. $d = 4$), we say that $\mathrm{Proj}_{\mathrm{nc}} S/(f)$ is a *noncommutative conic (resp. quadric)*.

Let $\mathrm{Sym}(3)$ be the symmetric group of degree 3 and

$$\mathrm{Sym}^3 V = \{w \in V^{\otimes 3} \mid \theta \cdot w = w \ \forall \theta \in \mathrm{Sym}(3)\}.$$

The following is one of the main results.

Theorem 19 ([5, Proposition 3.4, Lemma 3.5], [6, Corollary 3.8])**.** *Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$.*

(1) *If A is commutative, then*
   (a) *either $E = \mathbb{P}^2$ or $E \subset \mathbb{P}^2$ is a triple line, and*
   (b) *A is isomorphic to one of the following algebras:*

$$k[x, y, z]/(x^2),\ k[x, y, z]/(x^2 + y^2),\ k[x, y, z]/(x^2 + y^2 + z^2).$$

(2) *If A is not commutative, then*
   (a) *$|\sigma| = 2$, and*
   (b) *$S = \mathcal{D}(w, 1)$ for some $w \in \mathrm{Sym}^3 V$, and*

$$A \cong k\langle x, y, z\rangle/(yz + zy + ax^2, zx + xz + by^2, xy + yx + cz^2)/(\alpha x^2 + \beta y^2 + \gamma z^2)$$

for some $(a, b, c) \in k^3$ and $(\alpha, \beta, \gamma) \in \mathbb{P}^2$.

6. Classification of A

Lemma 20 ([6, Corollary 3.8])**.** *Let $S = \mathcal{A}(E, \sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then the quadratic dual algebra $A^! \cong k[X, Y, Z]/(F_1, F_2)$ is a complete intersection where $F_1, F_2 \in k[X, Y, Z]_2$.*

Lemma 21. *There are exactly 6 isomorphism classes of complete intersections of the form $k[X, Y, Z]/(F_1, F_2)$ where $F_1, F_2 \in k[X, Y, Z]_2$. (Classification of pencils of conics, see Table 1.)*

Table 1. List of $k[X, Y, Z]/(F_1, F_2)$

$k[X, Y, Z]/(X^2, Y^2)$,	$k[X, Y, Z]/(X^2 - YZ, Z^2)$,
$k[X, Y, Z]/(XZ + Y^2, YZ)$,	$k[X, Y, Z]/(X^2 - Y^2, Z^2)$,
$k[X, Y, Z]/(X^2 - YZ, Y^2 - XZ)$,	$k[X, Y, Z]/(X^2 - Y^2, X^2 - Z^2)$.

Corollary 22 ([6, Corollary 3.9]). *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. There are exactly 9 isomorphism classes of A (3 of them are commutative, and 6 of them are not commutative, see Table 2).*

TABLE 2. List of A

$k[x,y,z]/(x^2)$,	$k[x,y,z]/(x^2+y^2)$,	$k[x,y,z]/(x^2+y^2+z^2)$,
$S^{(0,0,0)}/(x^2)$,	$S^{(0,0,0)}/(x^2+y^2)$,	$S^{(0,0,0)}/(x^2+y^2+z^2)$,
$S^{(1,1,0)}/(x^2)$,	$S^{(1,1,0)}/(3x^2+3y^2+4z^2)$,	$S^{(1,1,0)}/(x^2+y^2-4z^2)$.

$S^{(a,b,c)} := k\langle x,y,z\rangle/(yz+zy+ax^2, zx+xz+by^2, xy+yx+cz^2)$.

7. Classification of E_A and $C(A)$

If S is a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$, then there exists a unique regular central element $f^! \in Z(A^!)_2$ such that $S^! = A^!/(f^!)$. We define $C(A) := A^![(f^!)^{-1}]_0$.

Theorem 23 ([12, Proposition 5.2]). *If S is a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ is a regular central element, and $A = S/(f)$, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\mathrm{mod}\, C(A))$.*

Theorem 24 ([5, Lemma 2.6], [6, Proposition 4.3, Lemma 4.4]). *Let $S = \mathcal{A}(E,\sigma)$ be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If A is not commutative, then the following holds:*

(1) *$C(A)$ is a 4-dimensional commutative Frobenius algebra.*
(2) *$Z(S)_2 = \{g^2 \mid g \in S_1\}$ (every $0 \neq f \in Z(S)_2$ is reducible!)*
(3) *A satisfies (G1). In fact, if $f = g^2$ for $g \in S_1$, then $\mathcal{P}(A) = (E_A, \sigma_A)$ where*

$$E_A = (E \cap \mathcal{V}(g)) \cup \sigma(E \cap \mathcal{V}(g)),\ \sigma_A = \sigma|_{E_A}.$$

Lemma 25 ([6, Proposition 4.3]). *If*

$$S = k\langle x,y,z\rangle/(yz+zy+ax^2, zx+xz+by^2, xy+yx+cz^2)$$

is a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$, then A is not commutative, and

$$\operatorname{Spec} C(A) \cong \{(\alpha,\beta,\gamma) \in \mathbb{A}^3 \mid (\alpha x+\beta y+\gamma z)^2 = f \ \text{ in } S\}/\sim$$

where $(\alpha,\beta,\gamma) \sim (-\alpha,-\beta,-\gamma)$.

Example 26. If $S = \mathcal{A}(E,\sigma) = k\langle x,y,z\rangle/(yz+zy, zx+xz, xy+yx)$ is a 3-dimensional Calabi-Yau quantum polynomial algebra, then

$$E = \mathcal{V}(xyz),$$
$$\begin{cases} \sigma(0,b,c) = (0,b,-c), \\ \sigma(a,0,c) = (-a,0,c), \\ \sigma(a,b,0) = (a,-b,0). \end{cases}$$

Further, if $f = x^2 + y^2 + z^2 \in Z(S)_2$, and $A = S/(f)$, then

$$(x + y + z)^2 = (x + y - z)^2 = (x - y + z)^2 = (x - y - z)^2 = f$$

in S and $C(A) \cong k^4$, so

$$\operatorname{Spec} C(A) \cong \{(1, 1, 1), (1, 1, -1), (1, -1, 1), (1, -1, -1)\} \subset \mathbb{A}^3.$$

Further, if $g = x + y + z$ so that $g^2 = f$, then

$$\begin{aligned} E_A &= \{(0, 1, -1), (-1, 0, 1), (1, -1, 0) \cup \sigma(\{(0, 1, -1), (-1, 0, 1), (1, -1, 0)\}) \\ &= \{(0, 1, -1), (-1, 0, 1), (1, -1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\} \subset \mathbb{P}^2. \end{aligned}$$

Theorem 27 ([6, Theorem 4.14])**.** *Let S, S' be 3-dimensional Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$ such that A, A' are not commutative. Then $E_A \cong E_{A'}$ if and only if $C(A) \cong C(A')$. There are exactly 6 isomorphism classes of E_A (see Table 3), so there are exactly 6 isomorphism classes of $C(A)$ (see Table 4). Moreover, every 4-dimensional commutative Frobenius algebra appears as $C(A)$.*

Table 3. Pictures of E_A when A is not commutative

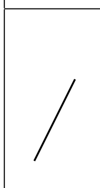
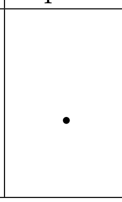
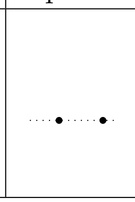
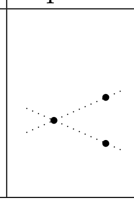
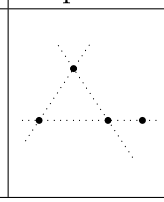
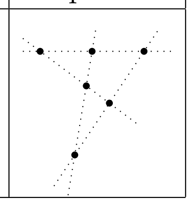
1 line	1 point	2 points	3 points	4 points	6 points
					

Table 4. List of $C(A)$ when A is not commutative

$k[u, v]/(u^2, v^2)$,	$k[u]/(u^4)$,	$k[u]/(u^3) \times k$,
$k[u]/(u^2) \times k[u]/(u^2)$,	$k[u]/(u^2) \times k^2$,	k^4.

Corollary 28. *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. There are exactly 9 isomorphism classes of $C(A)$.*

8. Classification of $\operatorname{Proj}_{\mathrm{nc}} A$

It is not easy to classify $\operatorname{Proj}_{\mathrm{nc}} A$ directly. Thanks to the classification of A and that of $C(A)$, we can complete the classification of $\operatorname{Proj}_{\mathrm{nc}} A$.

Theorem 29 ([6, Theorem 5.10])**.** *Let S, S' be 3-dimensional Calabi-Yau quantum polynomial algebras, $0 \neq f \in Z(S)_2, 0 \neq f' \in Z(S')_2$, and $A = S/(f), A' = S'/(f')$. Then*

$$A \cong A' \Rightarrow \operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj}_{\mathrm{nc}} A' \Rightarrow C(A) \cong C(A').$$

Corollary 30 ([6, Theorem 5.11])**.** *There are exactly 9 isomorphism classes of noncommutative conics in Calabi-Yau quantum $\mathbb{P}^2$'s.*

Finally, we focus on studying noncommutative smooth conics.

Definition 31. We say that $\mathrm{Proj}_{\mathrm{nc}} A$ is *smooth* if $\mathrm{gldim}(\mathrm{tails}\, A) < \infty$.

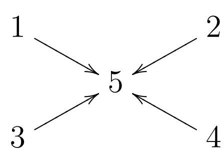
Theorem 32 ([12, Theorem 5.6], [11, Theorem 5.5])**.** *Let S be a d-dimensional quantum polynomial algebra, $f \in Z(S)_2$ a regular central element, and $A = S/(f)$. Then $\mathrm{Proj}_{\mathrm{nc}} A$ is smooth if and only if $C(A)$ is semisimple.*

Theorem 33 ([6, Theorem 5.15])**.** *Let S be a 3-dimensional Calabi-Yau quantum polynomial algebra, $0 \neq f \in Z(S)_2$, and $A = S/(f)$. If $\mathrm{Proj}_{\mathrm{nc}} A$ is smooth, then exactly one of the following two cases occur:*

(1) (a) *A is commutative.*
  (b) *f is irreducible.*
  (c) *$C(A) \cong M_2(k)$.*
  (d) *$\mathcal{D}^b(\mathrm{tails}\, A) \cong \mathcal{D}^b(\mathrm{mod}\, k\widetilde{A_1})$, where $k\widetilde{A}_1$ is the path algebra of the quiver*

$$1 \rightrightarrows 2 \qquad (\widetilde{A_1} \text{ type}).$$

(2) (a) *A is not commutative.*
  (b) *f is reducible.*
  (c) *$C(A) \cong k^4$.*
  (d) *$\mathcal{D}^b(\mathrm{tails}\, A) \cong \mathcal{D}^b(\mathrm{mod}\, k\widetilde{D_4})$, where $k\widetilde{D}_4$ is the path algebra of the quiver*



($\widetilde{D_4}$ type).

It is known that there are infinitely many Calabi-Yau quantum $\mathbb{P}^2$'s, so it is rather surprising that there are only 9 noncommutative conics in Calabi-Yau quantum $\mathbb{P}^2$'s up to isomorphism of noncommutative schemes, exactly two of them are smooth, and exactly one of them is irreducible.

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