SYMMETRIC COHOMOLOGY AND SYMMETRIC HOCHSCHILD COHOMOLOGY OF COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. Motivated by topological geometry, Staic defined the symmetric cohomology of groups by constructing an action of the symmetric group on the standard resolution which gives the group cohomology. In this paper, our aim is to construct the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras as a generalization of group algebras. In details, we will investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras. Also, we will investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras.

1. Introduction

This paper is based on [4]. Let G be a group and X a G-module. For $n \geq 0$, we set $C^n(G, X) = \{f : G^n \to X\}$. Motivated by topological geometry, Staic [5] defined the symmetric cohomology $\mathrm{HS}^\bullet(G, X)$ of a group G by constructing an action of the symmetric group $S_{\bullet+1}$ on the standard resolution $\mathrm{C}^\bullet(G, X)$ which gives the group cohomology $\mathrm{H}^\bullet(G, X)$. Also, Staic [6] studied the injectivity of the canonical map

$$\mathrm{HS}^\bullet(G, X) \to \mathrm{H}^\bullet(G, X)$$

induced by the inclusion $\mathrm{CS}^\bullet(G, X) \hookrightarrow \mathrm{C}^\bullet(G, X)$, where $\mathrm{CS}^\bullet(G, X) := \mathrm{C}^\bullet(G, X)^{S_{\bullet+1}}$ is the subcomplex of $\mathrm{C}^\bullet(G, X)$. Moreover, Staic [6] proved that the secondary symmetric cohomology group $\mathrm{HS}^2(G, X)$ is corresponding to extensions of groups which satisfies some conditions.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{C}^0(G,X) & \longrightarrow & \mathrm{C}^1(G,X) & \longrightarrow & \mathrm{C}^2(G,X) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{H}^\bullet(G,X) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & \mathrm{CS}^0(G,X) & \longrightarrow & \mathrm{CS}^1(G,X) & \longrightarrow & \mathrm{CS}^2(G,X) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HS}^\bullet(G,X)
\end{array}$$

In general, the cohomology of groups can be seen as the cohomology of group algebras. Recently, Coconet-Todea [1] defined the symmetric Hochschild cohomology $\mathrm{HHS}^\bullet(A, M)$ of twisted group algebras A which is a generalization of group algebras, where M is an A-bimodule.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathrm{C}^0_{\mathrm{e}}(A,M) & \longrightarrow & \mathrm{C}^1_{\mathrm{e}}(A,M) & \longrightarrow & \mathrm{C}^2_{\mathrm{e}}(A,M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HH}^\bullet(A,M) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
0 & \longrightarrow & \mathrm{CS}^0_{\mathrm{e}}(A,M) & \longrightarrow & \mathrm{CS}^1_{\mathrm{e}}(A,M) & \longrightarrow & \mathrm{CS}^2_{\mathrm{e}}(A,M) & \longrightarrow & \cdots & \Longrightarrow & \mathrm{HHS}^\bullet(A,M)
\end{array}$$

The detailed version of this paper has been submitted for publication elsewhere.

In this paper, our aim is to construct the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras as another generalization of group algebras. In details, we investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras (Theorem 6). Also, we investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras (Theorem 8).

2. Symmetric cohomology and symmetric Hochschild cohomology for cocomutative Hopf algebras

In the rest of this paper, let k be a field. For simplicity, we put $\otimes = \otimes_k$.

A k-algebra A is called *a Hopf algebra* if A is a k-algebra and a k-coalgebra satisfying

$$\pi \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \varepsilon = \pi \circ (S \otimes \mathrm{id}_A) \circ \Delta,$$

where the structure morphisms are as follows:

- $\pi : A \otimes A \to A$: product; $a \otimes b \mapsto ab$,
- $\eta : k \to A$: unit; $x \mapsto x \cdot 1_A$,
- $\Delta : A \to A \otimes A$: coproduct,
- $\varepsilon : A \to k$: counit,
- $S : A \to A$: antipode.

A Hopf algebra A is *cocommutative* if $a^{(1)} \otimes a^{(2)} = a^{(2)} \otimes a^{(1)}$ holds. Note that we use some standard notation for the coproduct, so called *Sweedler notation*; we write $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$, where the notation $a^{(1)}, a^{(2)}$ for tensor factors is symbolic. Throughout the paper, we omit the summation symbol $\sum$ of Sweedler notation when no confusion occurs (for details, see [7]).

Example 1. (1) Let G be a group, $A = kG$ a group algebra. For $g \in G$, we set
- coproduct $\Delta(g) := g \otimes g$,
- counit $\varepsilon(g) := 1$,
- antipode $S(g) := g^{-1}$,

then A is a cocommutive Hopf algebra.

(2) Let $A = k[X]$ be a polynomial ring. We set
- coproduct $\Delta(X) := 1 \otimes X + X \otimes 1$,
- counit $\varepsilon(X) := 0$,
- antipode $S(X) := -X$,

then A is a cocommutive Hopf algebra.

(3) Let A be a (cocommutaive) Hopf algebra. Then the opposite algebra A^{op} of A is a (cocommutative) Hopf algebra.

(4) Let A and B be (cocommutaive) Hopf algebras. Then $A \otimes B$ is a (cocommutative) Hopf algebra. In particular, if A is a (cocommutative) Hopf algebra, then the enveloping algebra $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ of A is a (cocommutative) Hopf algebra.

We recall the definition of a module over a Hopf algebra.

Definition 2 (cf. [9, Section 9.2])**.** Let A be a Hopf algebra and M, N left A-modules.

(1) For $a \in A$, $m \in M$ and $n \in N$,

$$a \cdot (m \otimes n) := a^{(1)} m \otimes a^{(2)} n. \text{ Then } M \otimes N \text{ is a left } A\text{-module.}$$

(2) For $a \in A$, $f \in \mathrm{Hom}_k(M, N)$ and $m \in M$,

$$(a \cdot f)(m) := a^{(1)} f(S(a^{(2)})m). \text{ Then } \mathrm{Hom}_k(M, N) \text{ is a left } A\text{-module.}$$

(3) A submodule ${}^A M$ of M is defined by ${}^A M := \{m \in M \mid a \cdot m = \varepsilon(a)m\}$, which is called an A-invariant submodule of M. For a right A-module M, M^A is defined similarly.

(4) Let M an A-bimodule. For $a \in A$ and $m \in M$, $a \cdot m := a^{(1)} m S(a^{(2)})$, which is called a left adjoint action. Using this action, we denote the left A-module by ${}^{\mathrm{ad}}M$. Similarly, we define a right adjoint action and M^{ad}.

Let A be a Hopf algebra, and M and N left A-modules. Then there is an isomorphism $\mathrm{Hom}_A(M, N) \cong {}^A(\mathrm{Hom}_k(M, N))$ as k-vector spaces (cf. [9, Lemma 9.2.2]). We define the cohomology of a Hopf algebra $\mathrm{H}^n(A, M) := \mathrm{Ext}^n_A(k, M)$.

Here, we construct the projective resolution of k as left A-modules as follows.

- $\widetilde{\mathrm{T}}_n(A) = A^{\otimes n+1}$; $\forall b \in A$,

$$b \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = b^{(1)} a_1 \otimes b^{(2)} a_2 \otimes \cdots \otimes b^{(n+1)} a_{n+1}.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{T}}}} \widetilde{\mathrm{T}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{T}}}} k \longrightarrow 0,$

$$d_n^{\widetilde{\mathrm{T}}}(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i) a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

We set the complex $\mathrm{K}^\bullet(A, M) := \mathrm{Hom}_A(\widetilde{\mathrm{T}}_\bullet(A), M)$.

Let A be a cocommutative Hopf algebra and M a left A-module. The n-th symmetric group is denoted by S_n. We define an action $\sigma_i = (i,\, i+1) \in S_{n+1}$ on $\mathrm{K}^n(A, M)$. For $f \in \mathrm{K}^n(A, M)$ $(1 \leq \forall i \leq n)$,

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+1}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}).$$

We set the subcomplex $\mathrm{KS}^\bullet(A, M) := \mathrm{K}^\bullet(A, M)^{S_{\bullet+1}}$ of $\mathrm{K}^\bullet(A, M)$.

Definition 3 ([4, Definition 3.3]). We define *the symmetric cohomology of a cocommutative Hopf algebra*

$$\mathrm{HS}^n(A, M) := \mathrm{H}^n(\mathrm{KS}^\bullet(A, M)),$$

Let A be a Hopf algebra and M an A-bimodule, where $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ is the enveloping algebra of A. We define Hochschild cohomology $\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, M)$ of A. We construct the projective resolution of A as A-bimodules as follows.

- $\widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) = A^{\otimes n+2}$; for $b \otimes c^{\mathrm{op}} \in A^{\mathrm{e}}$,

$$(b \otimes c^{\mathrm{op}}) \cdot (a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2}) = b^{(1)} a_1 \otimes b^{(2)} a_2 \otimes \cdots \otimes b^{(n+2)} a_{n+2} c.$$

- $\cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{T}}^{\mathrm{e}}}} \widetilde{\mathrm{T}}^{\mathrm{e}}_{n-1}(A) \longrightarrow \cdots \longrightarrow \widetilde{\mathrm{T}}^{\mathrm{e}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{T}}^{\mathrm{e}}}} A \longrightarrow 0,$

$$d_n^{\widetilde{\mathrm{T}}^{\mathrm{e}}}(a_1 \otimes \cdots \otimes a_{n+2}) = \sum_{i=1}^{n+1} (-1)^{i-1} a_1 \otimes \cdots \otimes \varepsilon(a_i) a_{i+1} \otimes \cdots \otimes a_{n+2}.$$

We set the complex $\mathrm{K}_{\mathrm{e}}^{\bullet}(A, M) := \mathrm{Hom}_{A^{\mathrm{e}}}(\widetilde{\mathrm{T}}^{\mathrm{e}}_{\bullet}(A), M)$.

Let A be a cocommutative Hopf algebra and M an A-bimodule. The n-th symmetric group is denoted by S_n. We define an action $\sigma_i = (i, i+1) \in S_{n+1}$ on $\mathrm{K}_{\mathrm{e}}^{n}(A, M)$. For $f \in \mathrm{K}_{\mathrm{e}}^{n}(A, M)$ $(1 \leq \forall i \leq n)$,

$$(\sigma_i \cdot f)(a_1 \otimes \cdots \otimes a_{n+2}) := -f(a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+2}).$$

We set the subcomplex $\mathrm{KS}_{\mathrm{e}}^{\bullet}(A, M) := \mathrm{K}_{\mathrm{e}}^{\bullet}(A, M)^{S_{\bullet+1}}$ of $\mathrm{K}_{\mathrm{e}}^{\bullet}(A, M)$.

Definition 4 ([4, Definition 3.8])**.** We define *the symmetric Hochschild cohomology of a cocommutative Hopf algebra*

$$\mathrm{HHS}^n(A, M) := \mathrm{H}^n(\mathrm{KS}_{\mathrm{e}}^{\bullet}(A, M)).$$

3. Main results

First, our aim is to investigate the relationships between the symmetric cohomology and the symmetric Hochschild cohomology for cocommutative Hopf algebras.

Theorem 5 ([2, Section 5])**.** *Let G be a group and X a G-bimodule. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HH}^n(\mathbb{Z}G, X) \cong \mathrm{H}^n(G, {}^{\mathrm{ad}}X)$$

as $\mathbb{Z}$-modules, where ${}^{\mathrm{ad}}X$ is a left G-module by $g \cdot x = gxg^{-1}$ for $g \in G$ and $x \in X$.

Theorem 5 is generalized to the case of Hopf algebras by Ginzburg-Kumar [3, Section 5].

For a cocomutative Hopf algebra, we have the following result which is a symmetric version of the classical results by Eilenberg-MacLane and Ginzburg-Kumar.

Theorem 6 ([4, Theorem 4.5])**.** *Let A be a cocommutative Hopf algebra and M an A-bimodule. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HHS}^n(A, M) \cong \mathrm{HS}^n(A, {}^{\mathrm{ad}}M)$$

as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action, that is, $a \cdot m = a^{(1)} m S(a^{(2)})$ for $m \in {}^{\mathrm{ad}}M$ and $a \in A$.

As a byproduct of Theorem 6, we have the following assertion.

Corollary 7 ([4, Corollary 4.6])**.** *Let A be a finite dimensional, commutative and cocommutative Hopf algebra. Then, for each $n \geq 0$, there is an isomorphism*

$$\mathrm{HHS}^n(A, A) \cong A \otimes \mathrm{HS}^n(A, k)$$

as k-vector spaces.

Secondly, our aim is to investigate the relationships between the cohomology and the symmetric cohomology for cocommutative Hopf algebras.

In [6] and [8], the following consequences were proved for the lower degree.

- $\mathrm{HS}^0(G, X) \cong \mathrm{H}^0(G, X)$.
- $\mathrm{HS}^1(G, X) \cong \mathrm{H}^1(G, X)$.
- $\mathrm{HS}^2(G, X) \hookrightarrow \mathrm{H}^2(G, X)$.

  Moreover, if G has no elements of order 2, then $\mathrm{HS}^2(G, X) \cong \mathrm{H}^2(G, X)$.

We consider the resolution of k;

- k is a trivial left kS_{n+1}-module; $\tau \cdot x = \varepsilon(\tau)x = x \quad (\tau \in S_{n+1},\ x \in k)$.
- $\widetilde{\mathrm{T}}_n(A)$ is a right kS_{n+1}-module; for $\sigma_i \in S_{n+1}$ $(1 \le \forall i \le n)$
$$(a_1 \otimes \cdots \otimes a_{n+1}) \cdot \sigma_i = -a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \otimes \cdots \otimes a_{n+1}.$$
- $\widetilde{\mathrm{S}}_n(A) := \widetilde{\mathrm{T}}_n(A) \otimes_{kS_{n+1}} k$.
- $\cdots \longrightarrow \widetilde{\mathrm{S}}_n(A) \xrightarrow{d_n^{\widetilde{\mathrm{S}}}} \cdots \longrightarrow \widetilde{\mathrm{S}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{S}}}} k \longrightarrow 0,$
$$d_n^{\widetilde{\mathrm{S}}}((a_1 \otimes \cdots \otimes a_{n+1}) \otimes_{kS_{n+1}} x) = d_n^{\widetilde{\mathrm{T}}}(a_1 \otimes \cdots \otimes a_{n+1}) \otimes_{kS_n} x.$$

Then we have the following isomorphism as complexes $\mathrm{KS}^\bullet(A, M) \cong \mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M)$. Therefore, we have $\mathrm{HS}^n(A, M) \cong \mathrm{H}^n(\mathrm{Hom}_A(\widetilde{\mathrm{S}}_\bullet(A), M))$.

Theorem 8 ([4, Theorem 4.9, Remark 4.10]). *Let A be a cocommutative Hopf algebra. For each $n \ge 1$, if $\mathrm{ch}\, k \nmid n+1$, then $\widetilde{\mathrm{S}}_n(A)$ is projective as a left A-module.*

Therefore, if $\mathrm{ch}\, k \nmid (n+1)!$, then, for each $0 \le m \le n$, there is an isomorphism
$$\mathrm{H}^m(A, M) \cong \mathrm{HS}^m(A, M)$$
as k-vector spaces.

Remark 9. (1) By Theorem 8, if $\mathrm{ch}\, k = 0$, then $\widetilde{\mathrm{S}}_\bullet(A)$ is a projective resolution of k, and hence there is an isomorphism $\mathrm{H}^\bullet(A, M) \cong \mathrm{HS}^\bullet(A, M)$ as k-vector spaces.
(2) Moreover, by Theorem 6 and Theorem 8, if $\mathrm{ch}\, k = 0$, then there is an isomorphism $\mathrm{H}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HS}^\bullet(A, {}^{\mathrm{ad}}M) \cong \mathrm{HHS}^\bullet(A, M)$ as k-vector spaces, where ${}^{\mathrm{ad}}M$ is a left A-module acting by the left adjoint action.

Finally, we give an example of the resolution which gives symmetric cohomology. Let p be an odd prime number, k a field of characteristic p and C_p a cyclic group of order p. Then we calculate the symmetric cohomology of $A = kC_p$.

Proposition 10 ([4, Proposition 4.11]). *Let p be an odd prime number, $\mathrm{ch}\, k = p$ and $A = kC_p$. Then $\widetilde{\mathrm{S}}_n(A)$ is a free A-module with rank $\dfrac{{}_pC_{n+1}}{p}$ for each $1 \le n \le p-2$.*

Since $\widetilde{\mathrm{S}}_{p-1}(A)$ is isomorphic to k as a left A-module, the resolution of k is the following exact sequence
$$0 \to k \xrightarrow{d_{p-1}^{\widetilde{\mathrm{S}}}} \widetilde{\mathrm{S}}_{p-2}(A) \to \cdots \to \widetilde{\mathrm{S}}_1(A) \xrightarrow{d_1^{\widetilde{\mathrm{S}}}} \widetilde{\mathrm{S}}_0(A) \xrightarrow{d_0^{\widetilde{\mathrm{S}}}} k \to 0,$$
where $\widetilde{\mathrm{S}}_i(A)$ is a free A-module for each $0 \le i \le p-2$. This implies that there is an isomorphism
$$\mathrm{H}^n(A, M) \cong \mathrm{HS}^n(A, M)$$
for any left A-module M and each $0 \le n \le p-2$. Also, in the case of $n = p-1$, the above isomorphism is obtained by simple calculation. Note that the period of the cohomology group $\mathrm{H}^n(A, M)$ of A is 2.

Summarizing the above, we have

$$\mathrm{HS}^n(A, M) \cong \begin{cases} \mathrm{H}^n(A, M) & (0 \leq n \leq p-1), \\ 0 & (p \leq n). \end{cases}$$

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