PROJECTIVE OBJECTS IN THE CATEGORY OF DISCRETE MODULES OVER A PROFINITE GROUP

RYO KANDA

Abstract. This is a summary of a joint work with Alexandru Chirvasitu. We showed that the category of discrete modules over an infinite profinite group has no non-zero projective objects and does not satisfy Ab4*.

Key Words: Profinite group; discrete module; projective object.

2010 *Mathematics Subject Classification*: Primary 18G05; Secondary 20E18, 16D40, 18E15, 18A30.

1. Introduction

This is a summary of a joint work with Alexandru Chirvasitu [CK19].

It is known that $\operatorname{Mod} R$ and $\operatorname{QCoh} X$ are both Grothendieck categories, where $\operatorname{Mod} R$ is the category of (left) modules over a ring R and $\operatorname{QCoh} X$ is the category of quasi-coherent sheaves on a scheme X. In particular, they both have exact direct limits (and hence exact direct sums) and enough injectives.

$\operatorname{Mod} R$ also has exact direct products (this property is called Grothendieck's Ab4* condition) and enough projectives, while it is known that none of these holds for $\operatorname{QCoh} X$ when X is a non-affine divisorial noetherian scheme:

Theorem 1 ([Kan19]). *Let X be a divisorial noetherian scheme. Then the following conditions are equivalent:*

(1) $\operatorname{QCoh} X$ *has enough projectives.*
(2) $\operatorname{QCoh} X$ *has exact direct products.*
(3) X *is an affine scheme.*

2. Main result

We consider a similar question concerning the category of discrete modules over a profinite group.

Definition 2. Let G be a topological group.

(1) G is called a *profinite group* if G is an inverse limit of finite discrete groups in the category of topological groups.

This is a summary of [CK19]. The detailed version of this paper has been submitted for publication elsewhere.

Ryo Kanda was supported by JSPS KAKENHI Grant Numbers JP16H06337, JP17K14164, JP20K14288, and JP21H04994, Leading Initiative for Excellent Young Researchers, MEXT, Japan, and Osaka Central Advanced Mathematical Institute: MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

(2) A *discrete G-module* is a topological G-module that is discrete as a topological space.

Let G be a profinite group. Considering a discrete G-module is equivalent to considering a (non-topological) G-module M (that is, left $\mathbb{Z}G$-module) such that the action $G \times M \to M$ is continuous if we endow M with the discrete topology. The forgetful functor gives an embedding of the category of discrete G-modules into $\operatorname{Mod}\mathbb{Z}G$ as a full subcategory. The essential image of the functor consists of all $M \in \operatorname{Mod}\mathbb{Z}G$ such that

$$M = \bigcup_{H} M^{H},$$

where H runs over all open normal subgroups of G and

$$M^{H} := \{\, x \in M \mid hx = x \text{ for all } h \in H \,\}.$$

Our main result is the following:

Theorem 3 ([CK19])**.** *Let G be a profinite group. Then the following conditions are equivalent:*

(1) *The category of discrete G-modules has enough projectives.*
(2) *The category of discrete G-modules has a nonzero projective object.*
(3) *The category of discrete G-modules has exact direct products.*
(4) *G is a finite group.*

References

[CK19] Alexandru Chirvasitu and Ryo Kanda, *Projective discrete modules over profinite groups*, arXiv:1907.12496v1.

[Kan19] Ryo Kanda, *Non-exactness of direct products of quasi-coherent sheaves*, Doc. Math. **24** (2019), 2037–2056. MR 4033823

Ryo Kanda
Department of Mathematics
Graduate School of Science
Osaka Metropolitan University
3-3-138, Sugimoto, Sumiyoshi, Osaka, 558-8585, Japan
Email address: `ryo.kanda.math@gmail.com`