ON THE OPENNESS OF LOCI OVER NOETHERIAN RINGS

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Abstract. In this article, we consider openness of loci of modules over commutative noetherian rings. One of the main theorems asserts that the finite injective dimension loci over an acceptable ring are open. We give a module version of the Nagata criterion, and confirm that it holds for some properties of modules.

Key Words: openness of loci, Nagata criterion, finite injective dimension, Gorenstein, Cohen–Macaulay.

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1. Introduction

We refer the reader to [3] (`arXiv:2201.11955`) for details on the contents of this article. Throughout this article, we assume that R is a commutative noetherian ring and that M is a finitely generated R-module.

Let $\mathbb{P}$ be a property of modules over a commutative local ring. The set of prime ideals $\mathfrak{p}$ of R such that the module $M_\mathfrak{p}$ over the local ring $R_\mathfrak{p}$ satisfies $\mathbb{P}$ is called the $\mathbb{P}$-locus of M (over R). There is a topology on $\operatorname{Spec}(R)$, which is called the Zariski topology. It is a natural question to ask when the $\mathbb{P}$-locus is open in the Zariski topology for a given $\mathbb{P}$. There are a lot of study studies about this question. The Cohen–Macaulay locus of a module over an excellent ring is open [2]. Furthermore, the Gorenstein locus of a module over an acceptable ring in the sense of Sharp [7] is open [4], and so is the finite injective dimension locus of a module over an excellent ring [8].

In this article, we consider the openness of the finite injective dimension locus of a module. The first main theorem of this article is the following theorem concerning the finite injective dimension locus over an acceptable ring.

Theorem 1. *The finite injective dimension locus of a module over an acceptable ring is open in the Zariski topology. In particular, the finite injective dimension locus of a module over a homomorphic image of a Gorenstein ring is open.*

For a property $\mathbb{P}$ of commutative local rings, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ satisfies $\mathbb{P}$ is called the $\mathbb{P}$-locus of R. Nagata [6] produced the following condition, which is called the Nagata criterion: if the $\mathbb{P}$-locus of $R/\mathfrak{p}$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of R, then the $\mathbb{P}$-locus of R is an open subset of $\operatorname{Spec}(R)$. This statement holds for the regular, complete intersection, Gorenstein, and Cohen–Macaulay properties and Serre's conditions; see [1, 5, 6]. We give a module version of the Nagata criterion for properties of modules, and show that it holds for the finite injective dimension property.

The detailed version [3] of this article has been submitted for publication elsewhere.

Theorem 2. *If the finite injective dimension locus of $M/\mathfrak{p}M$ over $R/\mathfrak{p}$ contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Supp}_R(M)$*, then the finite injective dimension locus of M over R is an open subset of* $\operatorname{Spec}(R)$*.*

It is seen that some results on the finite injective dimension property hold on other properties of modules; see [3].

2. Comments on Theorem 1

We begin with proving our key proposition. For an ideal I of R, we set $\mathrm{V}(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. Below is called the topological Nagata criterion.

Lemma 3. *Let U be a subset of* $\operatorname{Spec}(R)$*. Then U is open if and only if the following two statements hold true.*
(1) *If $\mathfrak{p} \in U$ and $\mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{q} \subseteq \mathfrak{p}$, then $\mathfrak{q} \in U$.*
(2) *U contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in U$.*

Denote by $\mathsf{FID}_R(M)$ the finite injective dimension locus of M over R. The Gorenstein locus of R is denoted by $\mathsf{Gor}(R)$. Note that $\mathsf{FID}_R(M)$ satisfies (1) in the above lemma for any R-module M. Therefore, in order to show that the finite injective dimension locus is open, it suffices to verify that it satisfies (2) in the above lemma. The key role is played by the proposition below.

Proposition 4. *Let $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \mathsf{FID}_R(M)$. The following conditions are equivalent.*
(1) *$\mathsf{FID}_R(M)$ contains a nonempty open subset of $\mathrm{V}(\mathfrak{p})$.*
(2) *$\mathsf{Gor}(R/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$.*

Proof. We may assume that for any integer $i \geq 0$, $\operatorname{Ext}^i_R(R/\mathfrak{p}, M)$ is free as an $R/\mathfrak{p}$-module. Let $I : 0 \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of M. The complex

$$0 \to \operatorname{Hom}_R(R/\mathfrak{p}, I^n) \xrightarrow{d^n} \operatorname{Hom}_R(R/\mathfrak{p}, I^{n+1}) \xrightarrow{d^{n+1}} \operatorname{Hom}_R(R/\mathfrak{p}, I^{n+2}) \xrightarrow{d^{n+2}} \cdots$$

is an injective resolution of $\operatorname{Ker} d^n$ as an $R/\mathfrak{p}$-module. For $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$, we see that $\mathfrak{q} \in \mathsf{FID}_R(M)$ if and only if $\mathfrak{q}/\mathfrak{p} \in \mathsf{FID}_{R/\mathfrak{p}}(\operatorname{Ker} d^n)$. It is easy to see that the latter holds if and only if $\mathfrak{q}/\mathfrak{p} \in \mathsf{Gor}(R/\mathfrak{p})$. This means that the equivalence holds. □

The result below can be obtained from Proposition 4.

Corollary 5. *Suppose that $\mathsf{Gor}(R/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\mathsf{FID}_R(M)$ is an open subset of $\operatorname{Spec}(R)$.*

Corollary 5 states that the finite injective dimension locus of M over R is open if the ring R satisfies the assumption of Nagata criterion for the Gorensteinness. Hence this corollary yields Theorem 1.

Remark 6. Theorem 1 recovers [8, Corollary 2.6] since any excellent ring is acceptable.

3. Comments on Theorem 2

Let $\mathbb{P}$ be a property of modules over a commutative local ring. In this section, we consider the following statement, which is the module version of the Nagata criterion: if the $\mathbb{P}$-locus of $M/\mathfrak{p}M$ over $R/\mathfrak{p}$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$, then the $\mathbb{P}$-locus of M over R is an open subset of $\operatorname{Spec}(R)$.

We denote by $\mathsf{Free}_R(M)$ the free locus of M over R. It is well-known fact that $\mathsf{Free}_R(M)$ is always open. We prepare the following lemma to state Theorem 2.

Lemma 7. *Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$. The following are equivalent.*
(1) $\mathsf{Gor}(R/\mathfrak{p})$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.
(2) $\mathsf{FID}_{R/\mathfrak{p}}(M/\mathfrak{p}M)$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.

Proof. We obtain $\operatorname{Supp}_R(M/\mathfrak{p}M) = \operatorname{V}(\mathfrak{p})$, and thus $\operatorname{Supp}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = \operatorname{Spec}(R/\mathfrak{p})$. Hence, we have

$$\mathsf{Gor}(R/\mathfrak{p}) \cap \mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M) = \mathsf{FID}_{R/\mathfrak{p}}(M/\mathfrak{p}M) \cap \mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M).$$

Since the set $\mathsf{Free}_{R/\mathfrak{p}}(M/\mathfrak{p}M)$ is a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$, we see that the equivalence holds. □

Theorem 2 follows from this lemma and Corollary 5. Theorem 2 asserts that the module version of the Nagata criterion holds for the finite injective dimension property.

4. Other properties of modules

Some results on the finite injective dimension property as we gave in the previous sections hold on other properties.

The Cohen–Macaulay locus of R is denoted by $\mathsf{CM}(R)$. Denote by $\mathsf{CM}_R(M)$ (resp. $\mathsf{MCM}_R(M)$) the Cohen–Macaulay (resp. the maximal Cohen–Macaulay) locus of M over R. The same assertion as Proposition 4 holds for the (maximal) Cohen–Macaulay property.

Proposition 8. *Let $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \mathsf{CM}_R(M)$. The following conditions are equivalent.*
(1) $\mathsf{CM}_R(M)$ *contains a nonempty open subset of* $\operatorname{V}(\mathfrak{p})$.
(2) $\mathsf{CM}(R/\mathfrak{p})$ *contains a nonempty open subset of* $\operatorname{Spec}(R/\mathfrak{p})$.
In addition, if $\mathfrak{p}$ belongs to $\mathsf{MCM}_R(M)$, then the following is also equivalent.
(3) $\mathsf{MCM}_R(M)$ *contains a nonempty open subset of* $\operatorname{V}(\mathfrak{p})$.

The result below is a Cohen–Macaulay version of Corollary 5.

Corollary 9. *Suppose that $\mathsf{CM}(R/\mathfrak{p})$ contains a nonempty open subset of $\operatorname{Spec}(R/\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Then $\mathsf{CM}_R(M)$ and $\mathsf{MCM}_R(M)$ are open.*

Remark 10. The same assertion as Corollary 5 holds for the Gorenstein property, the Cohen–Macaulay property, the maximal Cohen–Macaulay property, and Serre's conditions. In particular, the module version of the Nagata criterion holds for all of the aforementioned properties; see [3]. Those results recover theorems of Greco and Marinari [1] and of Massaza and Valabrega [5] about the Nagata criterion.

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