COMBINATORICS OF QUASI-HEREDITARY STRUCTURES

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Abstract. A quasi-hereditary algebra is a finite dimensional algebra together with a partial order on its set of isomorphism classes of simple modules which satisfies certain conditions. In this research, for a given algebra A, we study that how many partial orders make A to be quasi-hereditary. In particular, we classify such orders for path algebras of Dynkin type A. This proceeding is based on a paper [7].

1. Introduction

Quasi-hereditary algebras were defined in [11] as an algebraic axiomatization of the theory of rational representations of semi-simple algebraic groups. They were generalized to the concept of highest weight categories soon after in [2] as a tool to study highest weight theories which arise in the representation theories of semi-simple complex Lie algebras and reductive groups. There are many examples of such algebras, Schur algebras, algebras of global dimension at most two, incidence algebras and many more.

A quasi-hereditary algebra is a finite dimensional algebra together with a partial order on its set of isomorphism classes of simple modules which satisfies certain conditions. In the examples above, the partial order predated (and motivated) the theory, so the choice was clear (see [4]). However, there are instances of quasi-hereditary algebras having many possible choices of the partial order. So one may wonder about all the possible orders. In this research, we will study such all possible choices of orders.

Throughout this paper, let K be a field, A a finite dimensional K-algebra. We denote by $\mathsf{mod}A$ the category of finitely generated right A-modules and denote by $\{S(i) \mid i \in I\}$ a complete set of isomorphism classes of simple A-modules with an indexing set I. Let $P(i)$ and $I(i)$ be the projective cover and the injective envelop of $S(i)$, respectively. For an A-module M, we denote by $[M : S(i)]$ the Jordan-Hölder multiplicity of $S(i)$ in M.

Standard modules and costandard modules are fundamental concepts to define and study quasi-hereditary algebras.

Definition 1. Let $\lhd$ be a partial order on I. A *standard module* $\Delta(i)$ with weight $i \in I$ is the largest factor module of $P(i)$ such that each composition factor $S(j)$ satisfies $j \lhd i$. Dually, a *costandard module* $\nabla(i)$ with weight $i \in I$ is the largest submodule of $I(i)$ such that each composition factor $S(j)$ satisfies $j \lhd i$. We write $\Delta = \{\Delta(i) \mid i \in I\}$ and $\nabla = \{\nabla(i) \mid i \in I\}$.

Let Θ be a class of A-modules. We denote by $\mathcal{F}(\Theta)$ the subcategory of $\mathsf{mod}A$ consisting of all A-modules which have a Θ-filtration, that is, a module M with a chain of submodules $M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$ such that $M_i/M_{i-1} \in \Theta$.

The detailed version of this paper is [7].

Definition 2. [6] A partial order $\lhd$ on I is *adapted* to A if it satisfies that, for an A-module M with its top $S(i)$ and its socle $S(j)$, where i and j are incomparable by $\lhd$, there exists $k \in I$ such that $i \lhd k$ and $j \lhd k$ and $[M : S(k)] \neq 0$.

For example, any total order on I is adapted to A. In general, the standard, and the costandard modules will change when we refine the order. Dlab and Ringel [6] introduced adapted orders on I in order to consider refinements of partial orders. Namely, if $\lhd_2$ is a refinements of $\lhd_1$, then $\Delta_1(i) = \Delta_2(i)$ (also $\nabla_1(i) = \nabla_2(i)$) holds for any $i \in I$, where $\Delta_j(i)$ is a standard module with weight $i \in I$ associated to $\lhd_j$.

We define quasi-hereditary algebras.

Definition 3. [2, 6] Let $\lhd$ be a partial order on I. A pair $(A, \lhd)$ is *quasi-hereditary* if it satisfies the following statements.

(1) $\lhd$ is adapted to A.
(2) $[\Delta(i) : S(i)] = 1$ for any $i \in I$.
(3) $A \in \mathcal{F}(\Delta)$.

Quasi-hereditary algebras were introduced by Scott in [11] by using the existence of certain chain of ideals of A. In [2], Cline, Parshall and Scott gave a characterization of quasi-hereditary algebras by using highest weight categories and the existence of ∇-filtrations of injective modules. In their work, the order $\lhd$ on I was not assumed to be adapted, and the definition of quasi-hereditary algebras needs axioms which are different from the above.

For a partial order $\lhd$ on I, it is known by Conde [3] that if a pair $(A, \lhd)$ satisfies the axiom of quasi-hereditary algebras in [2], then the order $\lhd$ is adapted to A. Therefore assuming $\lhd$ to be adapted gives no restriction comparing with the definition in [2, 11].

For example, if A has global dimension at most two, then A is quasi-hereditary with some partial order. Any directed algebra is a quasi-hereditary algebra with some partial order. It is known that any quasi-hereditary algebra has finite global dimension.

We end this introduction to state the following characterization of hereditary algebras from the viewpoint of quasi-hereditary algebras.

Proposition 4. [5] *Let A be a finite dimensional K-algebra. Then $(A, \lhd)$ is quasi-hereditary for any adapted order $\lhd$ on I if and only if A is quasi-hereditary.*

2. Quasi-hereditary structures

To define quasi-hereditary structures on A, we need some notations. For an A-module T, let $T^{\perp}$ be a subcategory of $\mathsf{mod}A$ consisting of X such that $\mathrm{Ext}_A^i(T, X) = 0$ for all $i > 0$. Dually, we define ${}^{\perp}T$.

An A-module T is called a *tilting module* if T has finite projective dimension, $\mathrm{Ext}_A^i(T, T) = 0$ for all $i > 0$, and there exists an exact sequence with $T_i \in \mathsf{add}T$

$$0 \to A \to T_0 \to \cdots \to T_\ell \to 0.$$

We denote by $\mathsf{tilt}A$ the set of isomorphism classes of basic tilting A-modules. This set is a partially ordered set by $T_1 \leq_{\mathrm{tilt}} T_2$ if and only if $T_1^{\perp} \subseteq T_2^{\perp}$, see [8].

Theorem 5. [10] *Let $(A, \lhd)$ be a quasi-hereditary algebra. For each $i \in I$, there exists a unique indecomposable A-module $T(i)$ and short exact sequences*

$$0 \to \Delta(i) \to T(i) \to X(i) \to 0, \qquad 0 \to Y(i) \to T(i) \to \nabla(i) \to 0,$$

where $X(i)$ belongs to $\mathcal{F}(\Delta(j) \mid j \lhd i, j \neq i)$ and $Y(i)$ belongs to $\mathcal{F}(\nabla(j) \mid j \lhd i, j \neq i)$ such that

(1) *$T = \bigoplus_{i \in I} T(i)$ is a tilting A-module satisfying $\mathsf{add}T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.*
(2) *$\mathcal{F}(\Delta) = {}^{\perp}T$ and $\mathcal{F}(\nabla) = T^{\perp}$ hold.*

We say that the tilting module T in the above theorem the *characteristic tilting module* of $(A, \lhd)$. We have the following lemma.

Lemma 6. *Let $(A, \lhd_1)$ and $(A, \lhd_2)$ be quasi-hereditary algebras with basic characteristic tilting modules T_1, T_2, respectively. We denote by Δ_j the standard modules associated to $\lhd_j$ for $j = 1, 2$. The following statements are equivalent.*

(1) *$\Delta_1(i) = \Delta_2(i)$ for any $i \in I$.*
(2) *$\mathcal{F}(\Delta_1) = \mathcal{F}(\Delta_2)$.*
(3) *$T_1 \simeq T_2$.*

We are ready to define quasi-hereditary structures.

Definition 7. Let $(A, \lhd_1)$ and $(A, \lhd_2)$ be quasi-hereditary algebras with basic characteristic tilting modules T_1, T_2, respectively.

(1) We write $\lhd_1 \sim \lhd_2$ if $T_1 \simeq T_2$ holds.
(2) We denote by $\mathsf{qh.str}A$ the set of all equivalence classes of adapted orders to A defining A to be quasi-hereditary algebra modulo $\sim$ above, that is,

$$\mathsf{qh.str}A := \{\lhd \mid \lhd \text{ is an adapted order on } I,\ (A, \lhd) \text{ is quasi-hereditary}\}/\sim$$

We say that each element of $\mathsf{qh.str}A$ a *quasi-hereditary structure* of A. We denote by $[\lhd_1] \in \mathsf{qh.str}A$ the quasi-hereditary structure represented by $\lhd_1$.

(3) We write $[\lhd_1] \leq_{\mathrm{qh}} [\lhd_2]$ if $T_1 \leq_{\mathrm{tilt}} T_2$ holds. This gives a partial order on $\mathsf{qh.str}A$.

Note that for a quasi-hereditary structure $[\lhd]$, $\mathcal{F}(\Delta) = {}^{\perp}\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta)^{\perp} = \mathcal{F}(\nabla)$ hold [11]. Using this and Theorem 5, for quasi-hereditary structures $[\lhd_1], [\lhd_2] \in \mathsf{qh.str}A$, we have that $[\lhd_1] \leq_{\mathrm{qh}} [\lhd_2]$ if and only if $\mathcal{F}(\nabla_1) \subset \mathcal{F}(\nabla_2)$ if and only if $\mathcal{F}(\Delta_2) \subset \mathcal{F}(\Delta_1)$.

By Lemma 6, $(\mathsf{qh.str}A, \leq_{\mathrm{qh}})$ is a subposet of $(\mathsf{tilt}A, \leq_{\mathrm{tilt}})$. We study this partially ordered set. We first give some known results about $\mathsf{qh.str}A$.

Theorem 8. [4] *Let $(A, \lhd)$ be a quasi-hereditary algebra. Assume that there is a duality $F:\ \bmod A \to\ \bmod A$ such that $F(S(i)) \simeq S(i)$ for any $i \in I$ and $F^2 \simeq \mathrm{id}$. Then we have $|\mathsf{qh.str}(A)| = 1$.*

Since any refinement of an adapted order is also adapted, we have the following lemma.

Lemma 9. *We have $|\mathsf{qh.str}A| \leq |I|!$.*

Proof. Let $[\lhd] \in \mathsf{qh.str}A$ and $\lhd'$ a total order which is a refinement of $\lhd$. Then by the discussion [6, page 4] (see also [7, Lemma 2.3]), $\Delta(i) = \Delta'(i)$ holds for any $i \in I$. Namely, $\lhd \sim \lhd'$ holds. Therefore any quasi-hereditary structure is represented by a total order. We have the assertion. $\square$

There exists an algebra such that the above inequality is an equality.

Example 10. [7, Example 2.26] Let $\mathcal{C}_n$ be a quiver such that the set of vertices is $I = \{1, 2, \ldots, n\}$ and there is a unique arrow from i to j whenever $i > j$. In particular, the underlying graph of $\mathcal{C}_n$ is a complete graph. It is easy to see that any adapted order to $K\mathcal{C}_n$ is a total order on I, and two distinct total orders on I induce different quasi-hereditary structures on $K\mathcal{C}_n$. Therefore $|\mathsf{qh.str}(K\mathcal{C}_n)| = n!$ holds.

More precisely, one can show that $(\mathsf{qh.str}(K\mathcal{C}_n), \leq_{\mathrm{qh}})$ is isomorphic to the symmetric group S_n of rank n with the weak (Bruhat) order as partially ordered sets.

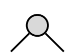
3. Quasi-hereditary structures of the path algebras of equioriented quivers of type A

Let $A_n = 1 \to 2 \to \cdots \to n-1 \to n$ be an equioriented A_n quiver. In this section, we see that $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ is isomorphic to $(\mathsf{tilt}(KA_n), \leq_{\mathrm{tilt}})$ as partially ordered sets. By definition, taking characteristic tilting module is an injective morphism of posets from $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ to $(\mathsf{tilt}(KA_n), \leq_{\mathrm{tilt}})$. To see that this is surjective, we use another description of tilting KA_n-modules via binary trees.

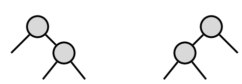
Binary trees can be defined inductively as follows. A *binary tree* T is either the empty set or a tuple $T = (r, L, R)$ where r is a singleton set, called the root of T, and L and R are two binary trees. The empty set has no vertex but has one leaf. The set of leaves of $T = (r, L, R)$ is the disjoint union of the set of leaves of L and R. The *size* of the tree is its number of vertices (equivalently the number of leaves minus 1).

The followings are the binary trees of size 1, 2 and 3:

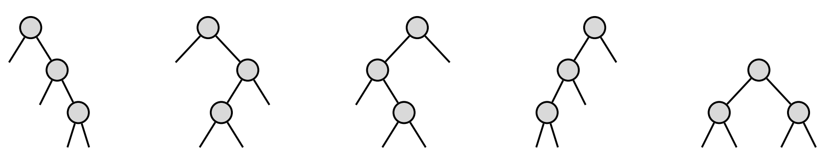
- binary tree of size 1:



- binary tree of size 2:



- binary tree of size 3:



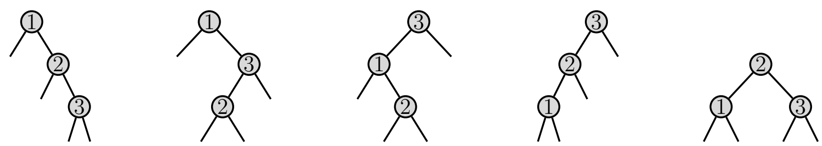
For each binary tree, there exists a unique labeling of its vertices, called a binary search tree, as follows.

Definition 11. A *binary search tree* of size n is a binary tree with n vertices labeled by $I = \{1, 2, \ldots, n\}$ with the following rule:

- if a vertex v is labeled by $k \in I$, then the vertices of the left subtree of v are labeled by integers less than k, and the vertices of the right subtree of v are labeled by integers superior to k.

This procedure is sometimes called the in-order traversal of the tree or simply as the in-order algorithm (recursively visit left subtree, root and right subtree).

Example 12. The binary search trees of size 3 are as follows.



We denote these binary search trees from left to right by $\mathbb{T}_i (i = 1, 2, \ldots, 5)$.

Since any binary tree admits a unique labeling of vertices making it to be a binary search tree, we always consider binary search trees.

Definition 13. For a binary search tree $\mathbb{T}$ of seize n, we define an order $\lhd_{\mathbb{T}}$ on $I = \{1, 2, \ldots, n\}$ by $i \lhd_{\mathbb{T}} j$ if and only if i labels a vertex of a subtree of a vertex labeled by j.

The following is one of main results of this study.

Theorem 14. [7] *Let $A_n = 1 \to 2 \to \cdots \to n-1 \to n$ be an equioriented* A_n *quiver and $I = \{1, 2, \ldots, n\}$. There exist bijections between the following three sets.*

(1) *The set of binary trees of size n.*
(2) *The set* $\mathsf{tilt}(KA_n)$.
(3) *The set* $\mathsf{qh.str}(KA_n)$.

In particular, we have that $|\mathsf{qh.str}(KA_n)|$ is equal to the Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

The bijection between (1) and (2) was shown by [1, Theorem 5.2], see also [9]. A bijection from (1) to (3) is given by $\mathbb{T} \mapsto \lhd_{\mathbb{T}}$. The point is that this map is well-defined and surjective. Therefore the natural map from $\mathsf{qh.str}(KA_n)$ to $\mathsf{tilt}(KA_n)$ is bijective. In particular, $(\mathsf{qh.str}(KA_n), \leq_{\mathrm{qh}})$ is isomorphic to $(\mathsf{tilt}(KA_n), \leq_{tilt})$ as partially ordered sets.

Note that each indecomposable KA_n-module is determined by its composition factors. For a given binary search tree $\mathbb{T}$, we have the composition factors of standard (resp. costandard) KA_n-modules $\Delta_{\mathbb{T}}(i)$ (resp. $\nabla_{\mathbb{T}}(i)$) associated to $\lhd_{\mathbb{T}}$ as follows.

Lemma 15. [7] *Let $\mathbb{T}$ be a binary search tree and $\Delta(i)$ (resp. $\nabla(i)$) the standard (resp. costandard) module associated to $\lhd_{\mathbb{T}}$. We denote by $T(i)$ the indecomposable direct summand of the characteristic tilting module of $(A, \lhd_{\mathbb{T}})$ as in Theorem 5.*

(1) *A simple module $S(j)$ is a composition factor of $\Delta_{\mathbb{T}}(i)$ (resp. $\nabla_{\mathbb{T}}(i)$) if and only if j labels a vertex in the right (resp. left) subtree of a vertex labeled by i.*
(2) *A simple module $S(j)$ is a composition factor of $T(i)$ if and only if j labels a vertex in either a left or a right subtree of a vertex labeled by i.*

Example 16. Let $\mathbb{T}_1, \ldots, \mathbb{T}_5$ be binary trees of size 3 as Example 12. For simplicity, we write $\lhd_{\mathbb{T}_i} = \lhd_i$. We denote by T_i the characteristic tilting modules of $(KA_n, \lhd_i)$. For each $\mathbb{T}_i$, we have

- $\lhd_1 = \{3 \lhd_1 2 \lhd_1 1\}$, $\quad T_1 = P(1) \oplus P(2) \oplus P(3)$.
- $\lhd_2 = \{2 \lhd_2 3 \lhd_2 1\}$, $\quad T_2 = P(1) \oplus S(2) \oplus P(2)$.
- $\lhd_3 = \{2 \lhd_3 1 \lhd_3 3\}$, $\quad T_3 = (P(1)/S(3)) \oplus S(2) \oplus P(1)$.

- $\lhd_4 = \{1 \lhd_4 2 \lhd_4 3\}$, $T_4 = S(1) \oplus (P(1)/S(3)) \oplus P(1)$.
- $\lhd_5 = \{1 \lhd_5 2, 3 \lhd_5 2\}$, $T_5 = S(1) \oplus P(1) \oplus S(3)$.

We have cover relatons $T_4 \leq_{\text{tilt}} T_3 \leq_{\text{tilt}} T_2 \leq_{\text{tilt}} T_1$ and $T_4 \leq_{\text{tilt}} T_5 \leq_{\text{tilt}} T_1$.

4. Concatenations of quivers and quasi-hereditary structures

Definition 17. Let Q^1, Q^2 be quivers and $v_i \in Q_0^i$ a sink. A *concatenation* of Q^1 and Q^2 at v_1 and v_2 is a quiver Q such that

- $Q_0 = (Q_0^1 \setminus \{v_1\}) \sqcup (Q_0^2 \setminus \{v_2\}) \sqcup \{v\}$
- $Q_1 = Q_1^1 \sqcup Q_1^2$, where we identify $v = v_1 = v_2$.

If $v_i \in Q_0^i$ is a source, we similarly define a concatenation at v.

Let Q be a concatenation of Q^1 and Q^2 at v. For a partial order $\lhd$ on Q_0, we have partial orders $\lhd|_{Q_0^1}$ on Q_0^1 and $\lhd|_{Q_0^2}$ on Q_0^2. Let $\overline{1} = 2$ and $\overline{2} = 1$. Conversely, we construct a partial order on Q_0 from partial orders on Q_0^ℓ. Let $\lhd^\ell$ be partial orders on Q_0^ℓ for $\ell = 1, 2$. Then we have a partial order $\lhd = \lhd(\lhd^1, \lhd^2)$ on Q_0 as follows: for $i, j \in Q_0$, $i \lhd j$ if one of the following two statements holds:

(1) $i, j \in Q_0^\ell$ and $i \lhd^\ell j$ holds for some ℓ,
(2) $i \in Q_0^\ell$, $j \in Q_0^{\overline{\ell}}$, $i \lhd^\ell v$ and $v \lhd^{\overline{\ell}} j$ hold.

We have the following theorem.

Theorem 18. [7] *Let Q be a concatenation of Q^1 and Q^2. Let A be a factor algebra of KQ and $A^i := A/\langle e_u \mid u \in Q_0 \setminus Q_0^i\rangle$ for $i = 1, 2$. Then we have an isomorphism of posets*

$$\mathsf{qh.str}(A) \longrightarrow \mathsf{qh.str}(A^1) \times \mathsf{qh.str}(A^2),$$

given by $[\lhd] \mapsto \big([\lhd|_{Q_0^1}], [\lhd|_{Q_0^2}]\big)$. The converse map is given by $(\lhd_1, \lhd_2) \mapsto \lhd(\lhd_1, \lhd_2)$.

Example 19. Let Q be a quiver $1 \to 2 \leftarrow 3$. This Q is a concatenation of $Q^1 = 1 \to 2$ and $Q^2 = 2 \leftarrow 3$ at 2. We have $\mathsf{qh.str}(KQ^1) = \{[1 \lhd 2], [2 \lhd 1]\}$ and $\mathsf{qh.str}(KQ^2) = \{[2 \lhd 3], [3 \lhd 2]\}$. So $|\mathsf{qh.str}(KQ)| = 4$ and we have

$$\mathsf{qh.str}(KQ) = \{[1 \lhd 2 \lhd 3], [1 \lhd 2, 3 \lhd 2], [2 \lhd 1, 2 \lhd 3], [3 \lhd 2 \lhd 1]\}.$$

Clearly, each path algebra of quivers of type A_n can be obtained by iterated concatenations of equioriented A_n quivers. So we can classify quasi-hereditary structures of such algebras.

Corollary 20. *Let Q be a quiver of type A_n obtained by iterated concatenations of $Q^1, Q^2, \ldots, Q^\ell$ such that each Q^i is an equioriented quiver of type A_{n_i} for some $n_i \in \mathbb{Z}_{\geq 1}$. Then there is a bijection*

$$\mathsf{qh.str}(KQ) \longrightarrow \prod_{i=1}^{\ell} \mathsf{qh.str}(KA_{n_i})$$

given by $[\lhd] \mapsto \big([\lhd|_{Q_0^i}]\big)_{i=1}^{\ell}$.

The bijection in Theorem 18 enables us to calculate characteristic tilting modules.

Let Q be a concatenation of Q^1 and Q^2 at a sink v. Let A be a factor algebra of KQ and $A^\ell := A/\langle e_u \mid u \in Q_0 \setminus Q_0^\ell\rangle$ for $\ell = 1, 2$. Fix two quasi-hereditary structures $[\lhd^\ell] \in \mathsf{qh.str}(A^\ell)$ and denote by $T^\ell(i)$ an indecomposable direct summands of the characteristic tilting module T^ℓ of $(A^\ell, \lhd^\ell)$ as in Theorem 5 for $i \in Q_0^\ell$. We denote by $\lhd = \lhd(\lhd^1, \lhd^2)$ the partial order on Q_0 as in Theorem 18. Let $T(i)$ be an indecomposable direct summands of the characteristic tilting module T of $(A, \lhd)$. Since v is a sink of Q^2, there is an injective morphism $S(v) \to T^2(v)$ by Theorem 5. Since A^ℓ is a factor algebra of A, we regard an A^ℓ-module as an A-module by a natural way.

Theorem 21. *Under the notation as above, for $i \in Q_0^1$, let m be the length of an $S(v)$-socle of $T^1(i)$. Then the push-out $U(i)$ of $T^2(v)^{\oplus m} \leftarrow S(v)^{\oplus m} \to T^1(i)$ is isomorphic to $T(i)$.*

Example 22. Consider Example 19. Put $\lhd = (1 \lhd 2 \lhd 3)$ which is an image of $([1 \lhd 2], [2 \lhd 3])$ by the map in Theorem 18. Then $T^1(1) = S(1)$ and $T^1(2) = P^1(1)$ are indecomposable direct summands of the characteristic tilting module of $(KQ^1, [1 \lhd 2])$. Also, $T^2(2) = S(2)$ and $T^2(3) = P^2(3)$ are indecomposable direct summands of the characteristic tilting module of $(KQ^2, [2 \lhd 3])$. By the above theorem, we have $T(1) = S(1)$, $T(2) = P(1)$ and $T(3) = I(2)$.

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