TILTING COMPLEXES OVER BLOCKS COVERING CYCLIC BLOCKS

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Abstract. Let p be a prime number, k an algebraically field of characteristic p, $\tilde{G}$ a finite group, and G a normal subgroup of $\tilde{G}$ having a p-power index in $\tilde{G}$. Moreover let B be a block of kG and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. In this note, we show that the set of isomorphism classes of basic tilting complexes over B is isomorphic to that of $\tilde{B}$ as partially ordered sets under some kinds of assumptions. Moreover, as an application, we give the result that the block $\tilde{B}$ of $k\tilde{G}$ covering a cyclic block is tilting-discrete block.

1. Introduction

In representation theory of finite groups, there is a well-known and important conjecture called Broué's abelian defect group conjecture.

Conjecture 1. *Let k be an algebraically closed field of characteristic $p > 0$, G a finite group, B a block of the group algebra kG with defect group D, and b the Brauer correspondent of B in $kN_G(D)$. If D is abelian, then the block B is derived equivalent to b.*

There are many cases that Broué's abelian defect group conjecture holds. Also, it is known that Broué's abelian defect group conjecture does not hold generally without the assumption that the defect group D is abelian. However even if the defect group D is not abelian, it is thought that the similar statement holds in some situations and that how we may state the non-abelian version conjecture. The one situation we are interested in is as follows: $\tilde{G}$ is a finite group with a normal subgroup G of p-power index in $\tilde{G}$ and G has a cyclic Sylow p-subgroup P. In fact, it is expected that the principal block $B_0(k\tilde{G})$ of $k\tilde{G}$ is derived equivalent to that $B_0(N_{\tilde{G}}(P))$ of $kN_{\tilde{G}}(P)$ (for example see [5]). To solve this, it is essential to find a suitable tilting complex over $B_0(k\tilde{G})$, but it is not easy. On the other hand, the study on tilting complexes over the principal block $B_0(kG)$ is well known and they have some kinds of good properties because the block $B_0(kG)$ is a cyclic block, which implies that it is a Brauer tree algebra (for example, see [1]). Based on these, we try to compare tilting complexes over $B_0(k\tilde{G})$ and those over $B_0(kG)$, and to give a classification of that over $B_0(k\tilde{G})$.

2. Silting theory

Let Λ be a finite dimensional algebra over an algebraically closed field. In [2], the set of isomorphism classes of basic silting complexes over Λ has a partially ordered set structure.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. [2] Let Λ be a finite dimensional algebra over an algebraically closed field. For silting complexes P and Q of $K^b(\operatorname{proj}\Lambda)$, we define a relation $\geq$ between P and Q as follows;

$$P \geq Q :\Leftrightarrow \operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(P, Q[i]) = 0 \ ({}^{\forall} i > 0).$$

Then the relation $\geq$ gives a partial order on $\operatorname{silt}\Lambda$, where $\operatorname{silt}\Lambda$ means the set of isomorphism classes of basic silting complexes over Λ.

Here, we remark that any silting complex over B is a tilting complex over B for any block algebra B of a finite group since it is symmetric algebra (for example, see [2, Example 2.8]). Hence, for a block algebra B of a finite group, the set of isomorphism classes of basic tilting complexes over B has a partially ordered set structure too. We denote this partially ordered set by $\operatorname{tilt} B$.

We recall the definition of mutations for silting complexes of $K^b(\operatorname{proj}\Lambda)$ [2, Definition 2.30, Theorem 3.1].

Definition 3. Let P be a basic silting complex of $K^b(\operatorname{proj}\Lambda)$ and decompose it as $P = X \oplus M$. We take a triangle

$$X \xrightarrow{f} M' \to Y \to$$

with a minimal left $(\operatorname{add} M)$-approximation f of X. Then the complex $\mu_X^-(P) := Y \oplus M$ is a silting complex in $K^b(\operatorname{proj}\Lambda)$ again. We call the complex $\mu_X^-(P)$ a left mutation of P with respect to X. If X is indecomposable, then we say that the left mutation is irreducible. We define the (irreducible) right mutation $\mu_X^+(P)$ dually. Mutation will mean either left or right mutation.

Remark 4. If B is a block algebra of finite group, then, for any tilting complex $P = X \oplus M$ over Λ, the complex $\mu_X^{\epsilon}(P)$ is a tilting complex again where ϵ means $+$ or $-$.

The following result is very important to study of partially ordered structure of the sets of silting complexes.

Theorem 5 ([2, Theorem 2.35])**.** *For any silting complexes P and Q over Λ, the following conditions are equivalent:*
(1) Q is an irreducible left mutation of P;
(2) P is an irreducible right mutation of Q;
(3) $P > Q$ and there is no silting complex L satisfying $P > L > Q$.

We recall the definition of tilting-discrete algebras.

Definition 6. We say that an algebra (which is not necessarily a symmetric algebra) Λ is a tilting-discrete algebra if for all $\ell > 0$ and any tilting complex P over Λ, the set

$$\{T \in \operatorname{tilt}\Lambda \mid P \geq T \geq P[\ell]\}$$

is a finite set.

Tilting-discrete algebras have the following nice property.

Theorem 7 ([3, Theorem 3.5])**.** *If Λ is a tilting-discrete algebra, then Λ is a strongly tilting connected algebra, that is, for any tilting complexes T and U, the complex T can be obtained from U by either iterated irreducible left mutation or iterated irreducible right mutation.*

3. Block theory

3.1. **Block theory.** In this section, let k be an algebraically closed field of characteristic $p > 0$. We denote by G a finite group, and by k_G the trivial module of kG, that is, a one-dimensional vector space on which each element in G acts as the identity. We recall the definition of blocks of group algebras. The group algebra kG has a unique decomposition

$$kG = B_1 \times \cdots \times B_n$$

into a direct product of subalgebras B_i each of which is indecomposable as an algebra. Then each direct product component B_i is called a block of kG. For any indecomposble kG-module M, there exists a unique block B_i of kG such that $M = MB_i$ and $MB_j = 0$ for all $j \in \{1, \ldots, n\} - \{i\}$. Then we say that M lies in the block B_i or that M is a B_i-module. Also we denote by $B_0(G)$ the principal block of kG, that is, the unique block of kG which does not annihilate the trivial kG-module k_G.

First, we recall the definition of defect groups of blocks of finite groups and their properties.

Definition 8. Let B be a block of kG. A minimal subgroup D of G which satisfies the following condition is uniquely determined up to conjugacy in G: the B-bimodule epimorphism

$$B \otimes_{kD} B \to B \ \ (b_1 \otimes_{kD} b_2 \mapsto b_1 b_2)$$

is a split epimorphism. We call the subgroup a defect group of the block B.

The following results are well known (for example, see [1]).

Proposition 9. *For the principal block $B_0(G)$ of kG, its defect group is a Sylow p-subgroup of G.*

Blocks with cyclic defect groups are called cyclic blocks. The cyclic blocks have good properties.

Proposition 10. *For a block B of kG and a defect group D of B, the following are equivalent:*

(1) D is a non-trivial cyclic group;
(2) B is of finite representation type and is not semisimple;
(3) B is a Brauer tree algebra.

We introduce the induced modules, induced complexes and covering blocks.

Definition 11. Let H be a subgroup of G. For a kH-module U, we denote by $\mathrm{Ind}_H^G U := U \otimes_{kH} kG$ the induced module of U from H to G. Also, for a complex $X = (X^i, d^i)$, we denote by $\mathrm{Ind}_H^G X$ the complex $(X^i \otimes_{kH} kG, d^i \otimes_{kH} kG)$. This induces a functor from $K^b(\mathrm{proj}\, kH)$ to $K^b(\mathrm{proj}\, kG)$.

Definition 12. Let G be a normal subgroup of $\tilde{G}$ and $\tilde{B}$ a block of $k\tilde{G}$. We say a block B of kG is covered by $\tilde{B}$ or $\tilde{B}$ covers B if there exists a non-zero $\tilde{B}$-module $\tilde{U}$ such that $\tilde{U}$ has a non-zero summand lying in B as a kG-module.

Let G be a normal subgroup of a finite group $\tilde{G}$. In general, for indecomposable kG-module U, the induced module $\mathrm{Ind}_G^{\tilde{G}}U$ is not indecomposable. Moreover, for a block B of kG, there are several blocks of $k\tilde{G}$ covering the block B. However, in case where G is a normal subgroup of a finite group $\tilde{G}$ and has a p-power index, the following propositions hold.

Proposition 13 ([4, Green's indecomposability theorem])**.** *If G is a normal subgroup of a finite group $\tilde{G}$ of p-power index, then for any indecomposable kG-module V the induced $k\tilde{G}$-module $\mathrm{Ind}_G^{\tilde{G}}V$ is an indecomposable $k\tilde{G}$-module.*

Proposition 14 ([6, Corollary 5.5.6])**.** *Let G be a normal subgroup of $\tilde{G}$, and B a block of G. If the index of G in $\tilde{G}$ is a p-power, then there exists a unique block of $k\tilde{G}$ covering B.*

Remark 15. Let G be a normal subgroup of a finite group $\tilde{G}$ of a p-power index, B a block of kG, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Then by Propositions 13 and 14, for any indecomposable complex X of $K^b(\mathrm{proj}\, B)$, we can easily show that the induced complex $\mathrm{Ind}_G^{\tilde{G}}X$ is an indecomposable complex of $K^b(\mathrm{proj}\, \tilde{B})$.

4. Main Results

In this section, we give our main results. Let G be a normal subgroup of $\tilde{G}$ with index in $\tilde{G}$ a p-power. First we give the tilting-discreteness of $\tilde{B}$ and an isomorphism between $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ as partially ordered sets, where B is a block of kG with some properties and $\tilde{B}$ is a unique block covering B.

Theorem 16. *Let $\tilde{G}$ be a finite group, G a normal subgroup such that the index $|\tilde{G} : G|$ is a p-power, k an algebraically closed field of characteristic $p > 0$, B a block of kG, and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Assume B satisfies the following conditions:*

(i) Any indecomposable B-module is $\tilde{G}$-invariant,
(ii) B is a tilting-discrete algebra,
(iii) Any algebra derived equivalent to B has a finite number of two-term tilting complexes.

Then $\tilde{B}$ is a tilting-discrete algebra. Moreover the induction functor $\mathrm{Ind}_G^{\tilde{G}} : K^b(\mathrm{proj}\, B) \to K^b(\mathrm{proj}\, \tilde{B})$ induces an isomorphism between $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ as partially ordered sets, here $\mathrm{tilt}\, B$ and $\mathrm{tilt}\, \tilde{B}$ mean the set of all tilting complexes over B and $\tilde{B}$ respectively.

As an application of above theorem, we can apply it to the case where we state in the introduction, that is, the case $\tilde{G}$ has a normal subgroup G with a p-power index in $\tilde{G}$ and with a cyclic Sylow p-subgroup. In fact, in this setting, the assumptions in Theorem 16 are satisfied automatically. Hence we get the following theorem.

Theorem 17. *Let $\tilde{G}$ be a finite group having G as a normal subgroup with index in $\tilde{G}$ a p-power. Let B be a block of the finite group G with cyclic defect group and $\tilde{B}$ the unique block of $k\tilde{G}$ covering B. Then the following hold.*

(1) $\tilde{B}$ is a tilting-discrete algebra.
(2) The induction functor $\mathrm{Ind}_G^{\tilde{G}} : \mathrm{tilt}\, B \to \mathrm{tilt}\, \tilde{B}$ induces an isomorphism of partially ordered sets.

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