RELATIVE STABLE EQUIVALENCES OF MORITA TYPE FOR THE PRINCIPAL BLOCKS OF FINITE GROUPS

NAOKO KUNUGI AND KYOICHI SUZUKI

Abstract. Wang and Zhang introduced the notion of stable equivalences of Morita type relative to pairs of modules for blocks of finite groups. First, in this paper, we give a method of constructing, in certain situations, relative stable equivalences of Morita type for the principal blocks of finite groups. Second, we introduce the notion of relative Brauer indecomposability, and give an equivalent condition for certain modules to be relatively Brauer indecomposable.

1. Introduction

Let G be a finite group, and k a filed of characteristic $p > 0$. We can decompose kG as a direct product of indecomposable k-algebras:

$$kG = B_1 \times \cdots \times B_n.$$

Each B_i is called a *block* of G. For any indecomposable kG-module U, there exists a unique block B_i such that $UB_i = U$. We write k_G for the *trivial kG-module*, that is, a one-dimensional k-vector space on which every element of G acts trivially. The group algebra kG has a unique block B such that $k_G B = k_G$, which is called the *principal block* of G and denoted by $B_0(G)$. We are interested in constructing Morita equivalences for the principal blocks of finite groups.

Broué [1] introduced the notion of stable equivalences of Morita type, and developed a method of constructing them for the principal blocks. This method has been used as one of the useful tools for constructing Morita equivalences for the principal blocks. However, we cannot use the method for finite groups having a common nontrivial central p-subgroup. On the other hand, Wang and Zhang [10] introduced the notion of relative stable equivalences of Morita type for blocks of finite groups, which is a generalization of stable equivalences of Morita type. In this paper, we state, as our first main theorem, a method for constructing relative stable equivalences of Morita type for the principal blocks.

In [5], the notion of Brauer indecomposability was introduced. The Brauer indecomposability of modules called Scott modules plays an important role in the method of Broué. Ishioka and the first author [4] gave an equivalent condition for Scott modules to be Brauer indecomposable. Although Brauer indecomposability of Scott modules is also useful for our first main theorem, somewhat more general condition is more appropriate. Therefore, in this paper, we introduce the notion of relative Brauer indecomposability, and state, as our second main theorem, an equivalent condition for Scott modules to be relatively Brauer indecomposable.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

In this section, we recall basic notation and definitions on modular representation theory and fusion systems.

Throughout this paper, we assume that k is an algebraically closed field of characteristic p, G is a finite group, and, unless other wise stated, modules are finitely generated right modules. We write $Z(G)$ for the center of G. Let H be a subgroup of G. We write $[H\backslash G]$ for a set of representatives of the right cosets of H in G. For a kG-module M, we write $M{\downarrow}_H^G$ for the restriction of M to H, and for a kH-module N, we write $N\uparrow_H^G = N \otimes_{kH} kG$ for the induced kG-module of N. For a kG-module M, we write $M^* = \mathrm{Hom}_k(M,k)$ for the k-dual of M, considered as a left kG-module.

For a p-subgroup Q of G, there is a unique indecomposable summand of $k_Q{\uparrow}^G$ such that it has k_G as a direct summand of the top. This indecomposable summand is called the *Scott kG-module* with vertex Q, and denoted by $S(G,Q)$.

Let M be a kG-module. For a subgroup H of G, we write M^H for the set of fixed points of H in M. For a p-subgroup Q of G, the *Brauer construction* of M with respect to Q is the $kN_G(Q)$-module $M(Q)$ defined as follows:

$$M(Q) = M^Q / \sum_R \mathrm{tr}_R^Q(M^R),$$

where R runs over the set of proper subgroups of Q, and $\mathrm{tr}_R^Q : M^R \to M^Q$, $\mathrm{tr}_R^Q(m) = \sum_{t\in[R\backslash Q]} mt$.

For subgroups H and K of G, we write

$$\mathrm{Hom}_G(H,K) = \{\varphi \in \mathrm{Hom}(H,K) \mid \varphi = c_g \text{ for some } g \in G \text{ such that } H^g \leq K\},$$

where c_g is a conjugation map. Let P be a p-subgroup of G. The *fusion system* of G over P is the category $\mathcal{F}_P(G)$ whose objects are the subgroups of P and morphisms are given by $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$. For subgroups Q and R of P, we say that Q and R are *$\mathcal{F}_P(G)$-conjugate* if Q and R are isomorphic in $\mathcal{F}_P(G)$. Let Q be a subgroup of P. We say that Q is *fully automized* in $\mathcal{F}_P(G)$ if $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$. We say that Q is *receptive* in $\mathcal{F}_P(G)$ if for any subgroup R of P and any $\varphi \in \mathrm{Iso}_{\mathcal{F}_P(G)}(R,Q)$, there is an element $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\bar{\varphi}|_Q = \varphi$, where $N_\varphi = \{g \in N_P(R) \mid {c_g}^{\varphi^{-1}} \in \mathrm{Aut}_P(Q)\}$. We say that Q is *fully normalized* in $\mathcal{F}_P(G)$ if $|N_P(Q)| \geq |N_P(R)|$ for any subgroup R of P that is $\mathcal{F}_P(G)$-conjugate to Q. The fusion system $\mathcal{F}_P(G)$ is *saturated* if any subgroup of P is $\mathcal{F}_P(G)$-conjugate to a subgroup that is fully automized and receptive.

3. Relative stable equivalences of Morita type

In this section, we first introduce results of Broué [1] and Linckelmann [7]. Next, we define the notion of relative stable equivalences of Morita type that was introduced by Wang and Zhang [10]. Finally, we state the first main theorem of this paper.

Broué [1] gave a method of constructing stable equivalences of Morita type:

Theorem 1. *(see* [1, Theorem 6.3]*) Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. If $(M(\Delta Q), M(\Delta Q)^*)$*

induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ for any nontrivial subgroup Q of P, then (M, M^*) induces a stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.

Linckelmann showed the following:

Theorem 2. *(see [7, Theorem 2.1]) Let B and B' be blocks of G and G', and M a B-B'-bimodule that is projective as a left module and a right module. Assume that $- \otimes_B M$ induces $\underline{mod}(B) \cong \underline{mod}(B')$. If for any simple B-module S, the B'-module $S \otimes_B M$ is simple, then $- \otimes_B M$ induces an equivalence between mod(B) and mod(B').*

Theorem 1 has been used as one of the useful tools for constructing Morita equivalences for principal blocks. In fact, we may construct a stable equivalences of Morita type by using Theorem 1, and lift it to a Morita equivalence by using Theorem 2. In this way, Morita equivalences has been confirmed in some cases, for example see [9] and [6]. However, we cannot use Theorem 1 if the common Sylow p-subgroup has a nontrivial subgroup Z that is a subgroup of $Z(G)$ and $Z(G')$. We see that $C_G(Z) = G$ and $C_{G'}(Z) = G'$. Hence we need to show that $B_0(G)$ and $B_0(G')$ are Morita equivalent in order to apply Theorem 1.

Okuyama [8] introduced the notion of projectivity relative to modules:

Definition 3. (see [8] and also [2, Section 8]) Let W be a kG-module. A kG-module M is *relatively W-projective* if M is a direct summand of $V \otimes W$ for some kG-module V.

Then we can define a relative stable category that is an analogue of the stable category. Let W be a kG-module. The *relative W-stable category* $\underline{\mathrm{mod}}^W(kG)$ of mod(kG) is the category whose objects are the finitely generated kG-modules, and whose morphisms are given by

$$\underline{\mathrm{Hom}}_{kG}^W(M, N) = \mathrm{Hom}_{kG}(M, N)/\mathrm{Hom}_{kG}^W(M, N),$$

where $\mathrm{Hom}_{kG}^W(M, N)$ is the subspace of $\mathrm{Hom}_{kG}(M, N)$ consisting of all homomorphisms that factor through a W-projective kG-module. Let B be a block of G. We write $\underline{\mathrm{mod}}^W(B)$ for the full subcategory of $\underline{\mathrm{mod}}^W(kG)$ whose objects are all B-modules. It follows that $\underline{\mathrm{mod}}^W(kG)$ has a structure of triangulated category (see [3, Theorem 6.2]), and $\underline{\mathrm{mod}}^W(B)$ is a triangulated subcategory of $\underline{\mathrm{mod}}^W(kG)$ (see [10, Proposition 3.1]).

Wang and Zhang [10] introduced the notion of relative stable equivalences of Morita type by using the notion of projectivity relative to modules:

Definition 4. (see [10, Definition 5.1]) Let G and G' be finite groups and B and B' blocks of G and G', respectively. For a kG-module W, a kG'-module W', a B-B'-bimodule M, and a B'-B-bimodule N, we say that the pair (M, N) induces a *relative (W, W')-stable equivalence of Morita type* between B and B' if M and N are finitely generated projective as left modules and right modules with the property that there are isomorphisms of bimodules

$$M \otimes_{B'} N \cong B \oplus X \quad \text{and} \quad N \otimes_B M \cong B' \oplus Y,$$

where X is $W^* \otimes W$-projective as a $k[G \times G]$-module and Y is $W'^* \otimes W'$-projective as a $k[G' \times G']$-module.

If $W = kG$, then it follows that X is projective as B-B-bimodule. Hence the notion of relative stable equivalences is a generalization of the notion of stable equivalences.

Finally, we state the first main theorem of this paper:

Theorem 5. *Let G and G' be finite groups with a common Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(G')$, and $M = S(G \times G', \Delta P)$. Assume that Z is a subgroup of P that is central in G and G'. If $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ for any subgroup Q of P properly containing Z, then (M, M^*) induces a relative $(k_Z{\uparrow}^G, k_Z{\uparrow}^{G'})$-stable equivalence of Morita type between $B_0(G)$ and $B_0(G')$.*

Note that Theorem 5 with $Z = 1$ implies Theorem 1 as $k_1{\uparrow}^G \cong kG$.

4. Relative Brauer indecomposability

In this section, we first recall from [5] and [4] the definition and some results of the Brauer indecomposability of kG-modules. Next, we introduce the notion of relative Brauer indecomposability and then state the second main theorem of this paper.

In [5], the notion of Brauer indecomposability was introduced:

Definition 6. (see [5]) A kG-module M is *Brauer indecomposable* if $M(Q)$ is indecomposable as $kQC_G(Q)$-module or zero.

In order to apply Theorem 1, the Scott module M must be Brauer indecomposable. In fact, for any nontrivial subgroup Q of P, we have to confirm that $(M(\Delta Q), M(\Delta Q)^*)$ induces a Morita equivalence between $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$, that is, the following bimodule isomorphisms hold:

$$M(\Delta Q) \otimes_{B_0(C_{G'}(Q))} M(\Delta Q)^* \cong B_0(C_G(Q)) \text{ and } M(\Delta Q)^* \otimes_{B_0(C_G(Q))} M(\Delta Q) \cong B_0(C_{G'}(Q)).$$

Since $B_0(C_G(Q))$ and $B_0(C_{G'}(Q))$ are indecomposable, the Brauer construction $M(\Delta Q)$ must be indecomposable as a $B_0(C_G(Q))$-$B_0(C_{G'}(Q))$-bimodule, or equivalently, as a $kC_{G \times G'}(\Delta Q)$-module. This means that M must be Brauer indecomposable.

Ishioka and the first author [4] gave conditions for Scott modules to be Brauer indecomposable:

Theorem 7. *(see* [4, Theorem 1.3]*) Let P be a p-subgroup of G, and $M = S(G, P)$. Suppose that the fusion system $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent:*

(i) The module M is Brauer indecomposable.

(ii) The module $S(N_G(Q), N_P(Q)){\downarrow}^{N_G(Q)}_{QC_G(Q)}$ is indecomposable for each fully normalized subgroup Q of P.

Moreover, if these conditions hold, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for any fully normalized subgroup Q of P.

Theorem 8. *(see* [4, Theorem 1.4]*) Let P be a p-subgroup of G, and Q a fully normalized subgroup of P. Assume that $\mathcal{F}_P(G)$ is saturated. If there exists a subgroup H_Q of $N_G(Q)$ satisfying the following conditions:*

(a) $N_P(Q)$ is a Sylow p-subgroup of H_Q,

(b) $|N_G(Q) : H_Q| = p^a, a \geq 0$,

then $S(N_G(Q), N_P(Q)){\downarrow}_{QC_G(Q)}$ is indecomposable.

Theorem 7 enables us to confirm the Brauer indecomposability of Scott modules by using the group-theoretic conditions in Theorem 8.

In Theorem 5, it suffices to show that $M(\Delta Q)$ is indecomposable as a $kC_{G\times G'}(\Delta Q)$-module for any subgroup Q of P properly containing Z while in Theorem 1, M must be Brauer indecomposable. Therefore we introduce the notion of relative Brauer indecomposability:

Definition 9. Let M be an indecomposable kG-module with vertex P, and R a subgroup of P. We say that M is *relatively R-Brauer indecomposable* if for any p-subgroup Q of G containing R, the Brauer construction $M(Q)$ is indecomposable (or zero) as a $kQC_G(Q)$-module.

Finally, we state the second main theorem of this paper:

Theorem 10. *Let P be a p-subgroup of G, and $M = S(G, P)$. Suppose that the fusion system $\mathcal{F}_P(G)$ is saturated, and Z is a subgroup of $Z(G) \cap P$. Then the following are equivalent.*

(i) The module M is relatively Z-Brauer indecomposable.

(ii) The module $S(N_G(Q), N_P(Q))\downarrow^{N_G(Q)}_{QC_G(Q)}$ is indecomposable for each fully normalized subgroup Q of P containing Z.

Moreover, if these conditions hold, then $M(Q) \cong S(N_G(Q), N_P(Q))$ for any fully normalized subgroup Q of P containing Z.

Note that Theorem 10 with $Z = 1$ implies Theorem 7. Theorem 10 enables us to confirm the relative Brauer indecomposability of Scott modules by using the group-theoretic conditions in Theorem 8.

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Department of Mathematics
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan
Email address: kunugi@rs.tus.ac.jp

Department of Mathematics
Graduate School of Science
Tokyo University of Science
1-3, Kagurazaka, Shinjuku-ku, Tokyo, 162-8601, Japan
Email address: 1119703@ed.tus.ac.jp