CATEGORICAL ENTROPY OF THE FROBENIUS PUSHFORWARD FUNCTOR

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Abstract. For a triangulated category $\mathcal{T}$ with a split generator G and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov-Haiden-Katzarkov-Kontsevich introduced the invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the categorical entropy. In this article, we will determine the categorical entropy of the Frobenius pushforward functor.

1. Introduction

This report is based on joint work with Ryo Takahashi [5].

For a triangulated category $\mathcal{T}$ and an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, Dimitrov, Haiden, Katzarkov, and Kontsevich [1] introduced the invariant $h_t^{\mathcal{T}}(\Phi)$ which is called the *categorical entropy* of Φ as a categorical analog of the topological entropy. The categorical entropy $h_t^{\mathcal{T}}(\Phi)$ is a function in one real variable t with values in $\mathbb{R} \cup \{-\infty\}$ and measures the complexity of the exact endofunctor Φ.

For a commutative noetherian local ring with prime characteristic p, the ring endomorphism $F : R \to R$, which is called the *Frobenius endomorphism*, is defined by $F(a) = a^p$. Assume further that $F : R \to R$ is module finite. The Frobenius endomorphism F induces two exact endofunctors: the *Frobenius pushforward*

$$\mathbb{R}F_* : \mathrm{D^b}(R) \to \mathrm{D^b}(R)$$

on the bounded derived category $\mathrm{D^b}(R)$ of finitely generated R-modules and the *Frobenius pullback*

$$\mathbb{L}F^* : \mathrm{K^b}(R) \to \mathrm{K^b}(R)$$

on the bounded homotopy category $\mathrm{K^b}(R)$ of finitely generated projective R-modules. Both these functors are the main tools to study singularities with positive characteristics.

For the Frobenius pullback functor $\mathbb{L}F^*$, Majidi-Zolbanin and Miasnikov [3] considered the full subcategory $\mathrm{K^b_{fl}}(R)$ of $\mathrm{K^b}(R)$ consisting of perfect complexes with finite length cohomologies and computed the categorical entropy $h_t^{\mathrm{K^b_{fl}}(R)}(\mathbb{L}F^*)$.

In this report, we study the Frobenius pushforward functor F_* and compute its categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}F_*)$. We will also discuss the relation between the categorical entropy $h_t^{\mathrm{D^b}(R)}(\mathbb{R}\phi_*)$ of the pushforward functor along a local ring endomorphism $\phi : R \to R$ and the *local entropy* $h_{loc}(\phi)$ of ϕ which has been introduced in [4].

The detailed version of this paper will be submitted for publication elsewhere.

2. Categorical Entropy

Let $\mathcal{T}$ be a triangulated category. We begin with fixing notations.

Notation 1. (1) For an object $X \in \mathcal{T}$, denote by $\mathrm{thick}(X)$ the smallest thick subcategory containing $\mathcal{T}$.

(2) For objects $X_1, X_2, \ldots, X_r \in \mathcal{T}$, we write $X_1 * X_2 * \cdots * X_r$ the subcategory of $\mathcal{T}$ consisting of objects $Y \in \mathcal{T}$ such that there are exact triangles

$$X_i \to Y_i \to Y_{i+1} \to X_i[1] \quad (i = 1, 2, \ldots, r-1)$$

with $Y_1 = Y$ and $Y_r = X_r$.

The following fact is basic:

Lemma 2. *The following conditions are equivalent for $X, Y \in \mathcal{T}$:*

(1) $Y \in \mathrm{thick}(X)$.

(2) *There are $Y' \in \mathcal{T}$, $n_1, n_2, \ldots, n_r \in \mathbb{Z}$ such that $Y \oplus Y' \in X[n_1] * X[n_2] * \cdots * X[n_r]$.*

Now, let us state the definitions of complexities and categorical entropies introduced in [1], which play central roles in this report.

Definition 3. (Dimitrov-Haiden-Katzarkov-Kontsevich)

(1) For $X, Y \in \mathcal{T}$ and $t \in \mathbb{R}$, define the *complexity* of Y relative to X by

$$\delta_t(X, Y) := \inf \left\{ \sum_{i=1}^{r} e^{n_i t} \;\middle|\; \begin{array}{c} \exists Y' \in \mathcal{T}, \exists n_1, n_2, \ldots, n_r \in \mathbb{Z} \text{ s.t.} \\ Y \oplus Y' \in X[n_1] * X[n_2] * \cdots * X[n_r] \end{array} \right\} \in [0, \infty].$$

By Lemma 2, $\delta_t(X, Y) < \infty$ if and only if $Y \in \mathrm{thick}(X)$.

(2) Assume that $\mathcal{T}$ has a split generator G (i.e., $\mathcal{T} = \mathrm{thick}(G)$). For an exact endofunctor $\Phi : \mathcal{T} \to \mathcal{T}$, define the *categorical entropy* of $(\mathcal{T}, \Phi)$ by

$$h_t^{\mathcal{T}}(\Phi) := \lim_{n \to \infty} \frac{1}{n} \log \delta_t(G, \Phi^n(G)).$$

This limit exists in $[-\infty, \infty)$ and is independent of the choice of G by [1, Lemma 2.6].

Here, we list basic properties of $\delta_t(X, Y)$.

Lemma 4. *Let $\mathcal{T}$ be a triangulated category.*

(1) *For $X, Y, Z \in \mathcal{T}$ with $Z \in \mathrm{thick}Y \subseteq \mathrm{thick}X$, one has $\delta_t(X, Z) \leq \delta_t(X, Y)\delta_t(Y, Z)$.*

(2) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X, Y) \leq \delta_t(X, Y \oplus Z) \leq \delta_t(X, Y) + \delta_t(X, Z)$.*

(3) *For $X, Y, Z \in \mathcal{T}$, one has $\delta_t(X \oplus Y, Z) \leq \delta_t(X, Z)$.*

(4) *For $X, Y \in \mathcal{T}$, one has $\delta_t(X, Y[n]) = \delta_t(X, Y)e^{nt}$.*

(5) *For $X, Y, Y_1, \ldots, Y_r \in \mathcal{T}$ with $Y \in Y_1 * \cdots * Y_r$, one has $\delta_t(X, Y) \leq \sum_{i=1}^{r} \delta_t(X, Y_i)$.*

(6) *For an exact functor $\Phi : \mathcal{T} \to \mathcal{T}'$ and $X, Y \in \mathcal{T}$, one has $\delta_t(\Phi(X), \Phi(Y)) \leq \delta_t(X, Y)$.*

3. Local and Categorical Entropies

Let $(R, \mathfrak{m}, k)$ be a d-dimensional commutative noetherian local ring. Let $\phi : R \to R$ be a finite local ring homomorphism.

Majidi-Zolbanin, Miasnikov, and Szpiro [4] defined the *local entropy* which measures the complexity of ϕ:

Definition 5. (Majidi Zolbanin-Miasnikov-Szpiro) Define the *local entropy* by

$$h_{loc}(\phi) := \lim_{n\to\infty} \frac{1}{n} \log(\mathrm{length}_R(R/\phi^n(\mathfrak{m})R)).$$

This limit exists and non-negative by [4, Theorem 1].

They determined the local entropy for the Frobenius homomorphism.

Proposition 6. ([4, Theorem 1]) *If R has prime characteristic p, then the equality*

$$h_{loc}(F) = d \log p$$

holds.

The aim of this report is to compare the local entropy of ϕ and the categorical entropy of exact endofunctors associated with ϕ. Let us recall two basic exact endofunctors. The pushforward functor

$$\phi_* : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$$

along ϕ is defined as follows: for an R-module M, $\phi_*(M) := M$ is an abelian group together with the R-module structure via ϕ. This functor is exact by definition. The pullback functor

$$\phi^* : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$$

along ϕ is defined by $\phi^*(M) := M \otimes_R \phi_*(R)$. Deriving these functors, we obtain exact functors

$$\begin{array}{cccc} \mathbb{R}\phi_* : & \mathrm{D^b}(R) & \longrightarrow & \mathrm{D^b}(R) \\ & \cup\!\shortmid & & \cup\!\shortmid \\ & \mathrm{D^b_{fl}}(R) & \longrightarrow & \mathrm{D^b_{fl}}(R), \end{array}$$

and

$$\begin{array}{cccc} \mathbb{L}\phi^* : & \mathrm{K^b}(R) & \longrightarrow & \mathrm{K^b}(R) \\ & \cup\!\shortmid & & \cup\!\shortmid \\ & \mathrm{K^b_{fl}}(R) & \longrightarrow & \mathrm{K^b_{fl}}(R). \end{array}$$

Here, $\mathrm{D^b_{fl}}(R)$ and $\mathrm{K^b_{fl}}(R)$ stand for the subcategories of $\mathrm{D^b}(R)$ and $\mathrm{K^b}(R)$ consisting of complexes with finite length cohomologies, respectively.

For the pullback functor $\mathbb{L}\phi^*$, Majidi-Zolbanin and Miasnikov compared the categorical entropy of $\mathbb{L}\phi^*$ and the local entropy of ϕ. Moreover, they determined the categorical entropy for the Frobenius pullback functor:

Theorem 7. *Let R be a d-dimensional commutative noetherian local ring and $\phi : R \to R$ a finite local ring homomorphism.*

(1) *For any $t \in \mathbb{R}$, one has the inequality*

$$h_t^{\mathrm{K^b_{fl}}(R)}(\mathbb{L}\phi^*) \geq h_{loc}(\phi).$$

(2) *Assume further that R is a complete noetherian local ring with prime characteristic p. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}^{\mathrm{pf}}_{\mathrm{fl}}(R)}(\mathbb{L}F^*) = h_{loc}(F) = d \log p$$

holds.

On the other hand, we can also compute $h_t^{\mathrm{K}^{\mathrm{b}}(R)}(\mathbb{L}F^*)$ and $h_t^{\mathrm{D}^{\mathrm{b}}_{\mathrm{fl}}(R)}(\mathbb{R}F^*)$.

Proposition 8. *Assume that R has prime characteristic and the Frobenius homomorphism $F : R \to R$ is finite. For any $t \in \mathbb{R}$, the following equalities hold:*

(1) $h_t^{\mathrm{K}^{\mathrm{b}}(R)}(\mathbb{L}F^*) = 0$.

(2) $h_t^{\mathrm{D}^{\mathrm{b}}_{\mathrm{fl}}(R)}(\mathbb{R}F^*) = \log[F_*(k) : k]$.

Remark 9. The triangulated categories $\mathrm{K}^{\mathrm{b}}_{\mathrm{fl}}(R), \mathrm{K}^{\mathrm{b}}(R), \mathrm{D}^{\mathrm{b}}_{\mathrm{fl}}(R)$ have generators $\mathrm{K}(\underline{x})$ (the Koszul complex of a system of generators $\underline{x}$ of $\mathfrak{m}$), R, k, respectively. Therefore the categorical entropies that appeared in the preceding results are defined.

From the above two results, the remained problem is to compute $h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}F_*)$, which we will consider in the next section.

4. Main Theorem

First note that if R is excellent, then the derived category $\mathrm{D}^{\mathrm{b}}(R)$ has a split generator and hence we can consider the categorical entropy $h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}\phi_*)$.

Theorem 10. *Let R be a d-dimensional excellent noetherian local ring.*

(1) *Let $\phi : R \to R$ be a finite local ring homomorphism. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}\phi_*) \geq h_{loc}(\phi) + \log[\phi_*(k) : k]$$

holds.

(2) *Assume further that R has prime characteristic p and the Frobenius homomorphism $F : R \to R$ is finite. For any $t \in \mathbb{R}$, the equality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(R)}(\mathbb{R}F_*) = h_{loc}(F) + \log[F_*(k) : k] = d \log p + \log[F_*(k) : k]$$

holds.

Using Theorem 10(1), we can globalize the inequality for the Frobenius pushforward functor:

Corollary 11. *Let X be a connected noetherian scheme with prime characteristic p. Assume that the Frobenius homomorphism $F : X \to X$ is finite. For any $t \in \mathbb{R}$ and $x \in X$, the inequality*

$$h_t^{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)}(\mathbb{R}F_*) \geq \dim \mathcal{O}_{X,x} \cdot \log p + \log[F_*(k(x)) : k(x)]$$

holds.

Remark 12. Since X is connected the number $\dim \mathcal{O}_{X,x} \cdot \log p + \log[F_*(k(x)) : k(x)]$ is independent of $x \in X$.

For the rest of this report, let us give a sketch of the proof of Theorem 10.

The proof of Theorem 10(1) needs the following lemma which generalizes [3, Lemma 2.1].

Lemma 13. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. Let $0 \neq G \in \mathrm{D^b}(R), 0 \neq P \in \mathrm{K^b_{fl}}(R)$ and take an integer N with $\mathrm{H}^i(G \otimes P) = 0$ for all $|i| > N$. Set $B := \max\{\mathrm{length}_R(\mathrm{H}^i(G \otimes P)) \mid -N \leq i \leq N\}$. Then for any $E \in \mathrm{D^b}(R)$, $m \in \mathbb{Z}$, and $t \in \mathbb{R}$, the inequality*

$$\delta_t(G, E) \geq B^{-1} e^{-mt} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^m(E \otimes_R P))$$

holds.

(Proof of Theorem 10(1)). We note that for a finitely generated R-module M with finite length, one has

$$\mathrm{length}_R((\phi^*)^n(M)) = [\phi^n(k) : k] \cdot \mathrm{length}_R(M) = [\phi(k) : k]^n \cdot \mathrm{length}_R(M).$$

In particular, $\mathrm{length}_R((\phi^*)^n(R)/\mathfrak{m}(\phi^*)^n(R))) = [\phi(k) : k]^n \cdot \mathrm{length}_R(R/\phi^n(\mathfrak{m})R)$.

Take a split generator $G \in \mathrm{D^b}(R)$ such that $\mathrm{H}^i(G) = 0$ for $i < 0$ and that R is a direct summand of $\mathrm{H}^0(G)$. Let $\underline{x}$ be a system of generators of $\mathfrak{m}$ and set $P = K(\underline{x})$ the Koszul complex of $\underline{x}$. Then it follows from Lemma 13 that

$$\begin{aligned}
\delta_t(G, (\mathbb{R}\phi_*)^n(G)) &\geq B^{-1} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^0((\mathbb{R}\phi_*)^n(G) \otimes_R P)) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R(\mathrm{H}^0((\mathbb{R}\phi_*)^n(G)) \otimes_R \mathrm{H}^0(P)) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(\mathrm{H}^0(G)) \cdot \otimes_R R/\mathfrak{m}) \\
&\geq B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(R) \otimes_R R/\mathfrak{m}) \\
&= B^{-1} e^{-N|t|} \cdot \mathrm{length}_R((\phi_*)^n(R/\phi^n(\mathfrak{m})R)) \\
&= B^{-1} e^{-N|t|} \cdot [\phi(k) : k]^n \cdot \mathrm{length}_R(R/\phi^n(\mathfrak{m})R)
\end{aligned}$$

Taking $\lim_{n\to\infty} \frac{1}{n} \log(-)$, we obtain $h_t(\mathbb{R}\phi_*) \geq h_{loc}(\phi) + \log[\phi(k) : k]$. □

Since the proof of Theorem 10(2), i.e., the proof of

$$h_t^{\mathrm{D^b}(R)}(\mathbb{R}F_*) \leq h_{loc}(F) + \log[\phi_*(k) : k]$$

is more difficult and complicated than the converse inequality, we shall give quite rough sketch of the proof. This difficulty comes from the fact that no explicit descriptions of a split generator of $\mathrm{D^b}(R)$ is known unlike $\mathrm{K^b_{fl}}(R), \mathrm{K^b}(R), \mathrm{D^b_{fl}}(R)$. Therefore, we use induction to reduce the case of $d = 0$ so that $\mathrm{D^b}(R) = \mathrm{D^b_{fl}}(R)$. The proof is done as follows:

- The case of $d = 0$ follows from Proposition 8(2).
- For the case of $d > 0$, first we reduce to domain case and then take a regular element x with $x\mathrm{Ext}_R^{2d+1}(-, -) = 0$. We can take such a regular element by [2, Theorem 5.3]. Then for a split generator G' of $\mathrm{D^b}(R/xR)$, $G := G' \oplus R$ is a split generator of $\mathrm{D^b}(R)$.

- Using Lemma 4, reduce to the computations of $\delta_t(G', (\mathbb{R}F_*)^n(G'))$, $\delta_t(G, (F_*)^n(R))$. $\delta_t(G', (\mathbb{R}F_*)^n(G'))$ is known by induction hypothesis. To compute $\delta_t(G, (F_*)^n(R))$, we use an exact sequence

$$0 \to \Omega^{2d}((F_*)^n(R)) \to R^{\oplus \beta_{2d-1}((F_*)^n(R))} \to \cdots \to R^{\oplus \beta_1((F_*)^n(R))} \to R^{\oplus \beta_0((F_*)^n(R))} \to (F_*)^n(R) \to 0$$

and the equality

$$\lim_{n\to\infty} \frac{1}{n} \log \beta_i((F_*)^n(R)) = d \log p + \log[F_*(k) : k];$$

see [6].

References

[1] G. Dimitrov, F. Haiden, L. Katzarkov, and M. Kontsevich, *Dynamical systems and categories*, Contemp. Math. **621** (2014), 133–170.

[2] S. B. Iyengar and R. Takahashi, *Annihilation of cohomology and strong generation of module categories*, Int. Math. Res. Not. IMRN **2** (2016), 499–535.

[3] M. Majidi-Zolbanin and N. Miasnikov, *Entropy in the category of perfect complexes with cohomology of finite length*, JPAA **223** (2019), no. 6, 2585–2597.

[4] M. Majidi-Zolbanin and N. Miasnikov, and L. Szpiro, *Entropy and flatness in local algebraic dynamics*, Publ. Mat. **57** (2013), 509–544.

[5] H. Matsui and R. Takahashi, *On the categorical entropy of the Frobenius pushforward functor*, `arXiv:2207.13774`.

[6] G. Seibert, *Complexes with homology of finite length and Frobenius functors*, J. Algebra **125** (1989), 278–287.

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